Notes on View Camera Geometry*

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Preface

The following is a collection of notes and derivations that I made while reading about view cameras. Everything is geometrical (trigonometrical), and thus may not apply in various ways to actual lenses; although it should be pointed out that while actual lens's may not attain these ideals, it is these for which lens designers strive.

1 Desargues's Theorem

There are three fundamental planes: the film plane, FP, the lens plane, LP, and the subject plane, SP. An object in SP will be in focus on FP only when the three planes bear a particular relationship. This relationship, interestingly enough, may be derived from a theorem of Desargues (1593-1662) in projective geometry.

Projective geometry is the name for that branch of mathematics that arose to explain the relationships of three dimensional objects, projected onto a two dimensional canvas; these relationships were observed by artists in the Renaissance (about 1400) when they began to develop an understanding of perspective. To draw a rug on a floor, for example, whose sides appear proper, one must choose a vanishing point, VP, on the horizon at which lines through the sides of the rug converge. The interesting thing is that if one draws a object, say box in three quarter view, then lines through its sides must converge to VP's on the horizon and all such VP's must lie on the same line.

This convergence to points on a line is the essence of Desargues's theorem.

> **Theorem (Desargues):** *If in a plane two triangles ABC and abc are situated so that the straight lines joining corresponding vertices are concurrent in a point O, then the corresponding sides, if extended, will intersect in three collinear points.*

This is illustrated in space by figure 1. The theorem is rather difficult to prove for triangles lying in the same plane, but it is quite easy to prove if they lie in different planes. In figure 1, aA, bB, cC are concurrent at O according to the hypothesis. Now ab lies in the same plane as AB, so that these two lines intersect at some point P2; likewise ac and AC intersect at P3, and cb and CB intersect at P1. Since P1, P2, and P3 are on extensions of the sides of abc and ABC, they lie in the same plane with each of these two triangles, and must consequently lie on the line of intersection of these two planes. Hence P1, P2, and P3 are collinear which was to be proved.

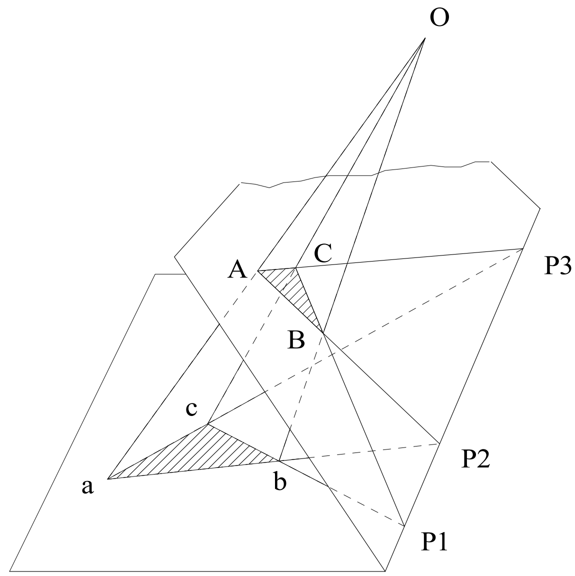


Figure 1: Desargues's configuration in space

It should be noted that the theorem holds, and the above proof follows through if one places the concurrent point O at infinity, as illustrated by figure 2, which brings us to Photography, since the three lines SP, LP, and FP in figure 2 represent the three photographic planes mentioned above. P1 may also be at infinity, in which case the three planes are parallel.

The point b in figure 2 is the lens center, and the lines P2a, and P3c are rays through the lens from SP to FP. The triangles P2Aa and P3Cc are those usually used to graphically depict image formation – see Stroebel, et.al. (1986). The lines aA and bB[1] and cC are drawn to be orthogonal to LP. Hence the extensions of aA, cC and bB are coincident at a point at infinity, satisfying the condition of Desargues's theorem, and thus the points P1, P2, and P3 are collinear. Since such a diagram may be drawn for any three lines which meet at a point, one has Scheimpflug's rule.

> **Scheimpflug's rule:** *The subject plane is rendered sharp when the subject plane, the lens plane, and the focus plane all meet in a line.*

[1]B may be placed anywhere on the line orthogonal to LP as far as Desargues's theorem is concerned, but for Photography, it should be placed one focal length away from b, making it the object focal point.

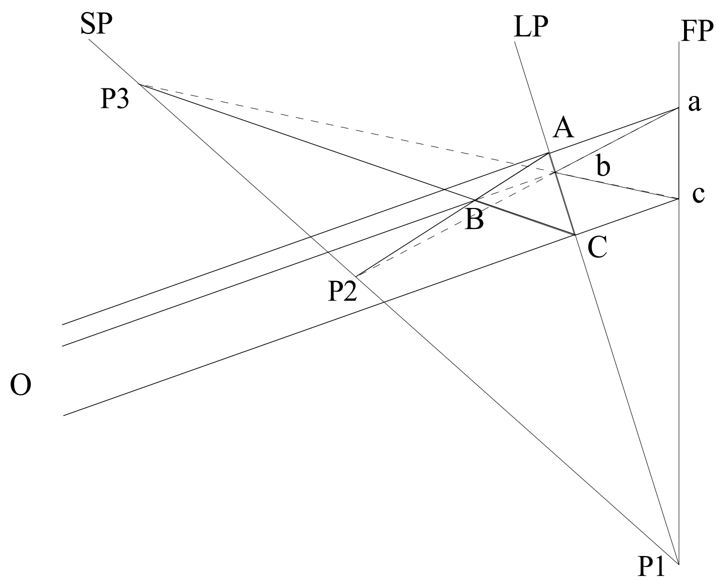


Figure 2: The three planes

2 The Gaussian Lens Equation

Figure 3 shows the three planes from figure 2 with some additional lines and points. The lines Aa, Bb and Cc are orthogonal to LP. The point L is the lens center, and the dashed line parallel to SP ending at point c is a line from infinity: since this line meets SP at infinity Desargues's theorem is satisfied. The length of Cc, f, is defined, as is customary, as the focal length of the lens. As has been noted, the three planes satisfy the Scheimpflug rule. We will now show that Desargues's theorem also implies the Gaussian lens equation:

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}. \tag{1}$$

Using figure 3, equation (1) may be obtained as follows:

$$\begin{aligned}
\frac{1}{u} + \frac{1}{v} &= \frac{1}{Pb\sin\alpha} + \frac{1}{Pa\sin\beta}, \\
&= \frac{1}{Pb\frac{f}{cL}} + \frac{1}{Pa\frac{f}{Pc}}, \\
&= \frac{1}{f}\left(\frac{cL}{Pb} + \frac{Pc}{Pa}\right),
\end{aligned}$$

and since by similar triangles $cL/Pb = ca/Pa$, one has

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}\left(\frac{ca}{Pa} + \frac{Pc}{Pa}\right) = \frac{1}{f}.$$

One can express this in terms of distances along the line a-b as

$$\frac{1}{h} + \frac{1}{y} = \frac{cos\omega}{f} \tag{2}$$

where $u = hcos\omega$ and $v = ycos\omega$.

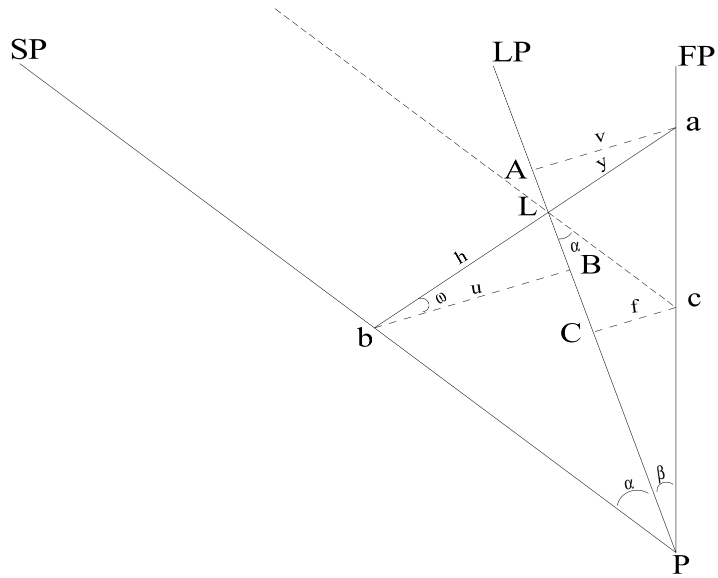


Figure 3: The lens equation

There are several derived forms of equation (1) that are frequently of use.

$$u = \frac{vf}{v-f}, \quad (3)$$

$$v = \frac{uf}{u-f}. \quad (4)$$

And if one defines magnification by

$$m = \frac{v}{u} = \frac{v-f}{f} = \frac{f}{u-f}, \quad (5)$$

then

$$u = f(1+\frac{1}{m}), \quad (6)$$

$$v = f(1+m), \quad (7)$$

and of course the Newtonian form:

$$f^2 = (u-f)(v-f). \quad (8)$$

These equations hold for h and y when f is replaced by $f/cos\omega$ as per equation(2). Later, use will be make of the fact that magnification, m, is constant for all values of ω. This follows algebraically and by inspection of the similar triangles in figure (3).

3 Thick lenses

The above is geometrical, and is an approximation to a thin lens, which is an abstraction describing a lens such that, within acceptable precision,the thickness can be ignored. Camera lenses are not thin. Fortunately, the consequences are negligible.

Light from a lens may be described geometrically as if it radiated from a single surface, which is essentially a plane for paraxial rays called the rear principal plane. This plane acts like a plane though a thin lens, and the focal length, distance v, and other variables on the lens side act as in the geometric theory above. The geometry is attributed to Gauss, see (Kingslake 1978) or (Mouroulis and Macdonald 1997). Because the lens is thick, variables on the object side are not quite what is given by the thin lens geometry, but the differences are usually so small as to have no effect on the acceptable precision of the calculations. The front principal plane is cognate to the rear principal plane in that they are images of one another with unit magnification. If δ is the nodal space[2], then u becomes $u' = u - \delta/\cos\alpha \approx u - \delta$, where α is the

[2]The nodal space or hiatus is the thickness of the lens as measured by the difference between the two nodal points. In air, the difference between the principal planes is equal to the nodal space.

angle of a ray with respect to the lens axis. hence for $u = 10$ meters, and $\delta = 2$ centimeters, the correction is $u' = 9.96$ meters. Even for close up photography, this difference hardly matters. It only becomes important when δ is substantial with respect to u – it should be noted that δ can be negative.

4 Pivot Points

Figure 4 shows the three planes and two lines from infinity: the line IQ meeting FP at Q and the line IP meeting SP at P. The distances from P and Q to LP must of course be f. J and K are the lengths of the rays from the lens center, L, to P and Q respectively. The points P and Q are pivot points, as will now be discussed.

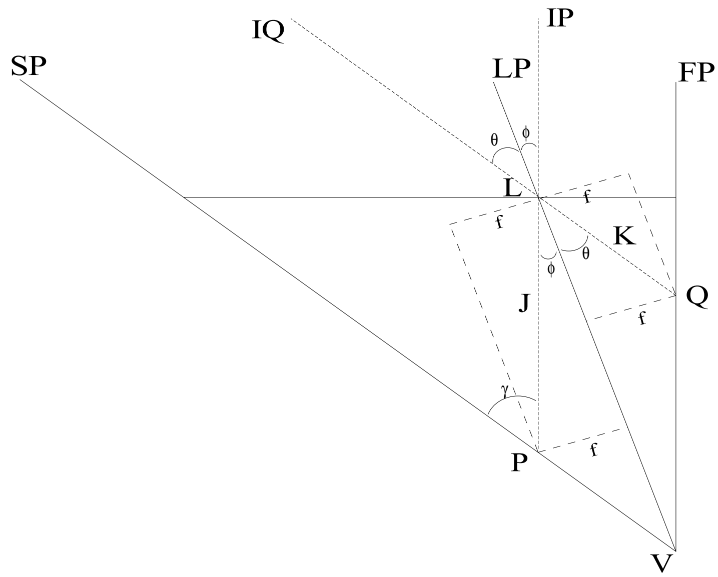


Figure 4: Pivot points

Suppose LP is fixed, and FP is moved parallel to itself, then SP must also move in order for the lens equation, equation (1), to be satisfied. SP does not move parallel to itself, as at first might be thought, nor does it pivot about V. It, in fact, turns about the *pivot* point P. P is the point where IP meets SP. By definition, P must be f units from LP, and since there is only one such point on the segment of IP below L, it follows that SP must pivot on P as FP is moved parallel to itself.

A similar statement may be made about the *reciprocal pivot* point Q with respect to movements of SP parallel to itself.

Later it will be seen that the near and far depth of field limits are essentially planes corresponding to movements of FP parallel to itself, hence they too must pivot on P.

The tilt angle of the lens, ϕ, with respect to FP is easily seen to be given by $\arcsin(f/J)$. Similarly the tilt with respect to SP is $\theta = \arcsin(f/K)$, and γ, the angle of SP with respect to FP, is $\phi + \theta$.

5 Determining the lens tilt

5.1 Using distances and angles

There are some authors, Merklinger(1993), who suggest that ϕ, the lens tilt angle, should be determined by guessing J. This can sometimes be done, since one does not need to know ϕ very precisely – within a degree is usually satisfactory. A more rigorous procedure is to measure the distance and angle of two points in SP, and then calculate J.

Consider figure 5 where two rays of length d and D have been drawn from the lens center on LP to SP. The longer ray is B degrees above the lens axis, of length Z, from the lens center and the shorter one is β degrees below the horizontal.

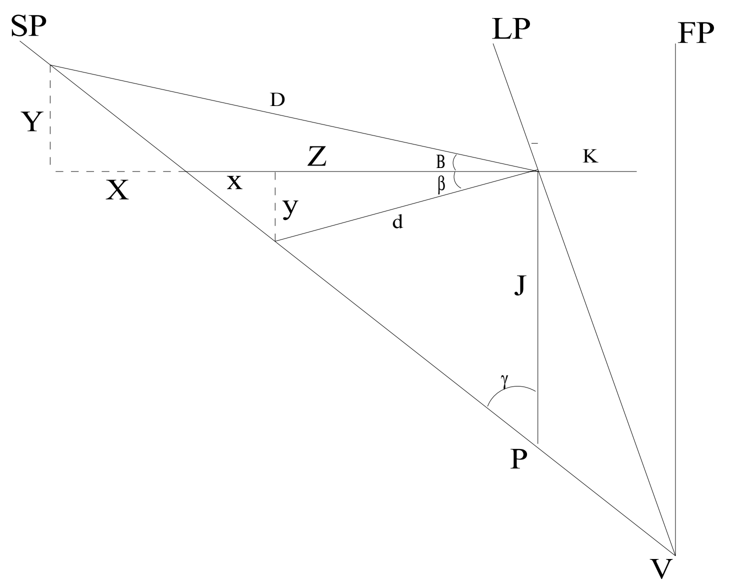


Figure 5: J from two rays

For the ray above the lens axis, one has:

$$\begin{aligned}
\frac{X}{Y} &= \frac{Z}{J}, \\
Y &= D\sin B, \\
X &= D\cos B - Z, \\
J &= \frac{ZD\sin B}{D\cos B - Z}, \\
Z &= \frac{JD\cos B}{D\sin B + J}.
\end{aligned}$$

Although not needed here, it is useful to note that $D = Z\cos(\gamma)/\cos(B+\gamma)$.

A similar calculation may be made for the ray below the lens axis, but one must adopt a convention about signs. There is a standard convention which chooses B negative and β positive. It seems better in the present case to reverse this convention; hence we will assume that angles below the lens axis are negative. This leads to a common formula for both types of rays:

$$J = \frac{Z\delta\sin\alpha}{\delta\cos\alpha - Z}, \tag{9}$$

$$Z = \frac{J\delta\cos\alpha}{\delta\sin\alpha + J}. \tag{10}$$

for an arbitrary ray of length δ and angle α.

If one can measure Z, then one may use equation (9) to find J and hence $\phi = \arcsin(f/J)$.

If one cannot, or chooses not to measure Z, then one may eliminate Z by using two rays and equating the values of equation (10) for each:

$$\begin{aligned}
\frac{Jd\cos\beta}{d\sin\beta + J} &= \frac{JD\cos B}{D\sin B + J}, \\
dD(\sin B\cos\beta - \cos B\sin\beta) &= J(d\cos\beta - D\cos B), \\
dD\sin(B-\beta) &= J(d\cos\beta - D\cos B), \\
\frac{1}{J} &= \frac{d\cos\beta - D\cos B}{dD\sin(B-\beta)}.
\end{aligned}$$

One may then find ϕ, using $\phi = \arcsin(f/J)$, from

$$\sin\phi = \frac{f(D\cos B - d\cos\beta)}{dD\sin(B-\beta)}. \tag{11}$$

The angle, γ, SP makes with respect FP will be needed shortly. It is given by $\gamma = \arctan(Z/J)$. If Z is known, then from equation (9), $\tan\gamma = (\delta\cos\alpha - Z)/\delta\sin\alpha$. If Z is not known, then one may find Z by equating the values from equation 9 for two rays, obtaining:

$$Z = \frac{dD \sin(B - \beta)}{D \sin B - d \sin \beta},$$

and then,

$$\tan \gamma = \frac{D \cos B - d \cos \beta}{d \sin \beta - D \sin B}. \tag{12}$$

The distance K shown in figure 5, is used in subsection 5.2. It is the conjugate of Z for an untilted lens:

$$K = \frac{f}{Z - f}.$$

5.2 Using back focus

Figure 6 shows both a tilted lens plane (TLP), and a tilted film plane (TFP). Two points, A and C, on SP are shown in focus on TFP at a and c respectively.

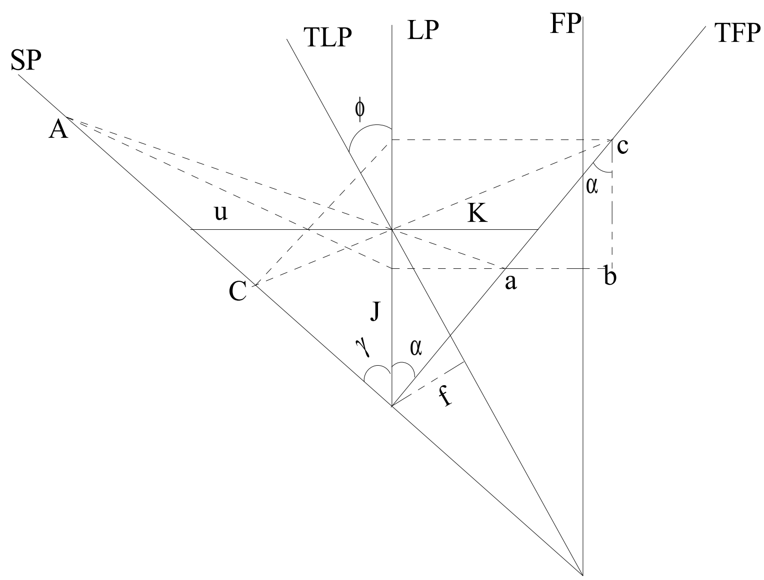


Figure 6: Back movements to obtain tilt

Note that the conjugates of SP points are on both TFP and on vertical planes through the conjugates. Thus the distance from A to LP is conjugate to the distance from LP to a, and these distances are the same both for a tilted film plane and for a vertical film plane. Thus one may locate a and c by moving FP. The angle α is given as the arctangent of $(b - a)/(c - b)$, and it is easily seen that ϕ may be found from

$$\sin(\phi) = \frac{f}{K}\tan\alpha = Q\sin(\alpha),$$

where

$$Q = \frac{f}{K\cos(\alpha)} = \frac{1}{(1+m')\cos(\alpha)},$$

where $K = uf/(u-f) = f(1+m')$ with m' the magnification for a ray through the untilted lens center. The distance from the lens center to FP is of course $uf/(u\cos(\phi) - f)$ because ω in equation (2) is ϕ. See figure (20). In general Q is not greatly different from unity, hence one may often take $\phi = \alpha$.

Finally, one may find γ from

$$\tan(\gamma) = \frac{Kf}{K-f}\frac{1}{J} = \frac{f}{K-f}\tan(\alpha) = \frac{1}{m'}\tan(\alpha);$$

which leads to:

$$\sin(\phi) = \frac{1}{\frac{1}{\tan(\alpha)} + \frac{1}{\tan(\gamma)}}.$$

In other words $\sin(\phi)$ is half the harmonic mean of $\tan(\alpha)$ and $\tan(\gamma)$.

5.3 Wheeler's rules

A more practical expression for use in the field is Wheeler's rule of 60, given by

$$\phi \approx 60\frac{(b-a)}{(c-b)}\frac{(u-f)}{u},$$

where $(u-f)/u$ is close to unity except for close-up work.

Since some cameras do not have angle scales, one can calculate the tilt in terms of the distance Δ that the top of the front standard moves. Let T be the distance between the top of the front standard and the tilt axis, then Wheeler's tilt rule is

$$\Delta = T\frac{(b-a)}{(c-b)}\frac{(u-f)}{u},$$

where $(u-f)/u$ is close to unity except for close-up work.

Some cameras do not have movable backs, and must be focused by moving the lens. The rules may still be used – but not for close up work.

> EXAMPLE: A plane containing a nearby flower and a distance mountain sweeps from under the photographer's feet passing through both. The rail distance between the two points of focus on the camera is 12 mm. On the groundglass, the two images are about 60

mm apart. Wheeler's rule of 60 indicates that the lens should be tilted by 60*12/60=12 degrees. If the height of the front standard measured from the tilt axis to the top of the standard is 25 cm, then Wheeler's tilt rule indicates that lens should be tilted forward until the top of the standard has moved forward 25*12/60=5 cm.

5.4 Lens Movement

Figure (6) places the center of the lens at the same point for both a tilted and untilted lens. But in fact the lens center is not at the same point for the two situations. The lens moves as it is tilted. It moves slightly with center tilt cameras, and substantially with base tilt cameras. This has little effect on the calculation of ϕ since if the lens center in figure (6) is moved horizontally by Δ, one has $\phi = \arcsin(f/(J-\theta))$, where $\theta = \Delta/\tan(\gamma)$, which is usually a negligible change from $\phi = \arcsin(f/J)$ because Δ is seldom more than about 50 millimeters, and J is in meters.

Lens movement has more effect on the focus position, since FP, for a tilted lens, is at a horizontal distance of $uf/(u\cos(\phi) - f)$ from the lens center, which is larger than $K = uf/(u-f)$ for the untilted lens. For a base tilt camera, the FP for a tilted lens will be at approximately $K - \Delta$, hence usually in front of a. For a center tilt camera, the FP will usually be in back of a.

5.5 Back Tilts

Tilting the lens preserves perspective because the magnification defined by v/u remains unchanged. If h and y are distances as shown in figure (3), then $y/h = (y\cos\omega)/(h\cos\omega) = v/u$ for any tilt, and thus tilting leaves magnification unchanged. On the other hand, tilting the back changes the v/u ratio, and the perspective. Parallel lines can be made to meet at vanishing points and *vice versa*.

When the back is tilted, it is also necessary to tilt the lens to bring the subject into focus. Figure (7) shows the relevant planes, where LP and FP are the untilted lens and focal planes, and TB is the tilted back plane. The back is tilted α degrees, and the desired lens tilt is given by $\alpha + \beta$. It may be seen that

$$D = \frac{J\sin\gamma}{\sin(\gamma-\alpha)}, \tag{13}$$

and thus β is given by

$$\beta = \arcsin\left(\frac{f\sin(\gamma-\alpha)}{J\sin\gamma}\right) = \arcsin\left(\frac{t}{m+1}\frac{\sin(\gamma-\alpha)}{\sin\gamma}\right),$$

where $t = H/V$ in figure (7), and m is the magnification as in figure (6).

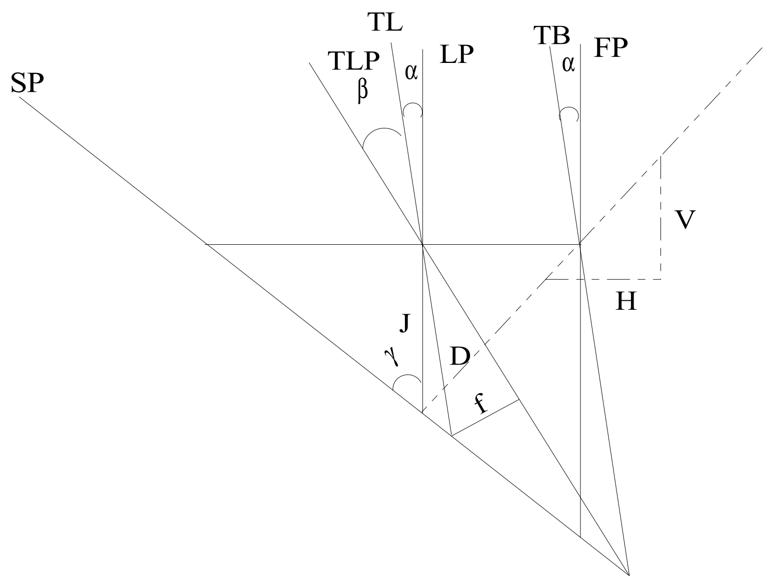


Figure 7: Lens tilt for tilted back

6 Depth of field for parallel planes

Figures 8 and 9 show diagrams that will be used in deriving depth of field, DOF, equations for the case when SP, LP, and FP are parallel. The diameters of the aperture and circle of confusion are d and c, respectively.

6.1 Near DOF limit

For figure 8, the circle of confusion arises because the image plane, IP is behind FP. The near DOF distance is given by equation (3) as

$$Z_n = \frac{(v+x)f}{(v+x)-f}.$$

By similar triangles, one sees that $x = cv/(d-c)$, hence the near DOF limit is

$$Z_n = \frac{vdf}{vd + f(d-c)} = \frac{uf^2}{f^2 + (u-f)Nc} = \frac{u(H-f)}{H+u-2f}, \tag{14}$$

where $N = f/d$ is the f-number, and

$$H = \frac{f^2}{Nc} + f \tag{15}$$

is the *hyperfocal* distance.

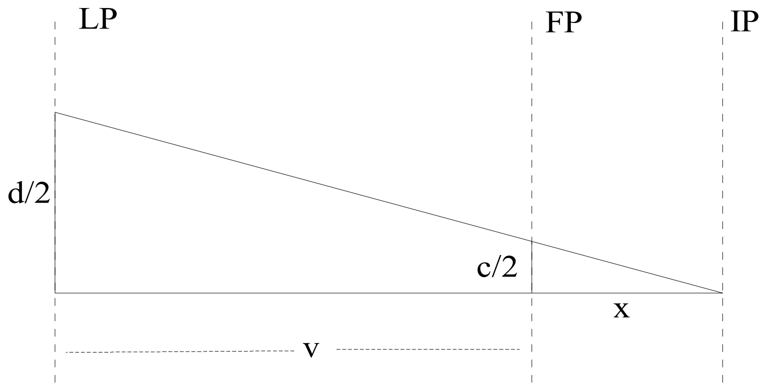


Figure 8: DOF near

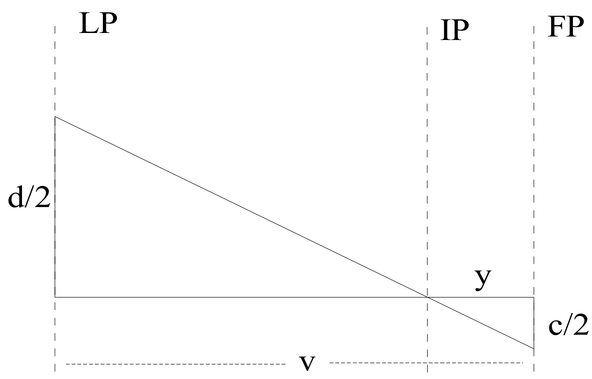


Figure 9: DOF far

The *Front DOF* is $u - Z_n = u(u-f)/(H+u-2f)$.
If one takes $H_a = \frac{f^2}{Nc}$, then one has

$$Z_n = \frac{uH_a}{H_a + (u-f)}. \tag{16}$$

One may also recast these equations in terms of the distance, t, from the focal plane. Since $t = u+v+\delta$, where δ is the nodal space, one has $t = \delta + u^2/(u-f)$ and $u(t) = (t-\delta)\left(\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{f}{t-\delta}}\right)$. It follows that the near distance from the focal plane is

$$Z_{nt} - \delta = \frac{u(t)f^2}{f^2 + (u(t)-f)Nc} + u(t)f/(u(t)-f) = \frac{u(t)f}{u(t)-f}\frac{u(t)f + (u(t)-f)Nc)}{f^2 + (u(t)-f)Nc}.$$

or perhaps more succinctly

$$Z_{nt} - \delta = Z_n \left(\frac{u(t)}{(u(t)-f)} + \frac{N}{f} \right) = Z_n \left(\frac{t}{f(1+1/m)} + \frac{Nc}{f} \right) = Z_n \left(\frac{t}{f(1+1/m)} + \frac{f}{H_a} \right),$$

where m is magnification.

6.2 Far DOF limit

For figure 9, IP is in front of FP, giving

$$u' = \frac{(v-y)f}{(v-y)-f}.$$

By similar triangles, one has $y = cv/(d+c)$, and

$$u' = \frac{vdf}{vd - f(d-c)} = \frac{uf^2}{f^2 - (u-f)Nc}.$$

When y is zero, the denominator vanishes because the rays from the object are parallel, hence the far DOF limit is

$$Z_f = \begin{cases} \frac{uf^2}{f^2-(u-f)Nc} = \frac{u(H-f)}{H-u} & \text{for } u < H \\ \infty & \text{for } u \geq H \end{cases} \tag{17}$$

The *Back DOF* is $Z_f - u = u(u-f)/(H-u)$ for $u < H$ and infinity otherwise. If one takes $H_a = \frac{f^2}{Nc}$, then for $(u-f) < H_a$ one has

$$Z_f = \frac{uH_a}{H_a - (u-f)}. \tag{18}$$

In terms of the distance, t, from the focal one has for $u(t) < H$

$$Z_{ft} - \delta = \frac{u(t)f^2}{f^2 - (u(t)-f)Nc} + u(t)f/(u(t)-f) = \frac{u(t)f}{u(t)-f} \frac{u(t)f - (u(t)-f)Nc)}{f^2 - (u(t)-f)Nc},$$

or perhaps more succinctly

$$Z_{ft} - \delta = Z_f \left(\frac{u(t)}{(u(t)-f)} - \frac{Nc}{f} \right) = Z_f \left(\frac{t}{f(1+1/m)} - \frac{Nc}{f} \right) = Z_f \left(\frac{t}{f(1+1/m)} - \frac{f}{H_a} \right).$$

6.3 DOF

The DOF is the difference $Z_f - Z_n$, so

$$DOF = \left\{ \begin{array}{ll} \frac{2Nc(u-f)uf^2}{f^4-[Nc(u-f)]^2} = \frac{2u(u-f)(H-f)}{(H-f)^2-(u-f)^2} & \text{for } u < H \\ \infty & \text{for } u \geq H \end{array} \right), \tag{19}$$

or

$$DOF = \left\{ \begin{array}{ll} \frac{2Ncvf}{(v-f)^2-(Nc)^2} = \frac{2vf^3(H-f)}{(v-f)^2(H-f)^2-f^4} & \text{for } v < Hm \\ \infty & \text{for } v \geq Hm \end{array} \right),$$

An interesting variant of this may be obtained by substituting equation (6) into equation (19):

$$DOF = \left\{ \begin{array}{ll} \frac{2Nc\frac{m+1}{m^2}}{1-K} & \text{for } K < 1, \\ \infty & \text{for } K \geq 1 \end{array} \right. \tag{20}$$

where $K = (\frac{Nc}{mf})^2$, which for small K (as might occur in macro photography) becomes

$$DOF \approx 2Nc\frac{m+1}{m^2} = 2Nc\left(\frac{S}{I} + \left(\frac{S}{I}\right)^2\right), \tag{21}$$

where S is the subject height, and I the image height.

In general, one should use

$$DOF = \frac{2Nc(m+1)}{m^2 - (\frac{Nc}{f})^2}. \tag{22}$$

This equation shows how DOF decreases as f increases. It is interesting to note that the denominator of equation 22 vanishes when u equals the hyperfocal distance as given by equation 15. In practice, this means that DOF depends on f when $u > H/2$, say, but not for closer subjects.

The near and far limits in terms of m are:

$$Z_n = \frac{f(1+m)}{m + \frac{Nc}{f}}, \tag{23}$$

and

$$Z_f = \left\{ \begin{array}{ll} \frac{f(1+m)}{m-\frac{Nc}{f}} & \text{for } mf > Nc \\ \infty & \text{for } mf \leq Nc \end{array} \right. \tag{24}$$

6.4 Circles of confusion

The human eye has a resolution limit of about 5 line pairs per millimeter (l/mm) at the nearest distance of distinct vision (about 25 cm for a normal eye). This means that a normal human eye can just discriminate between finely drawn lines separated by 0.2 mm. One can check this for oneself with a lens chart. It follows that 5 l/mm is the maximum resolution needed by a print. The field of view of human vision is about 60 degrees, which corresponds to about 290 mm at 25 cm. The diagonal of an 8x10 print is 325 mm, and viewing such a print at 25 cm completely fills the field of view. For such a print viewed at such a distance, the perspective will be correct if the photograph was taken with a standard lens[3]. This means that to preserve perspective, one should view a 16/20 print at 50 cm, and other prints proportionally.

For an 8x10 contact print, one should have at least 5 l/mm. When the image size is smaller, magnification comes into play: thus there will be a factor of 8 for a 35 mm image. This implies geometrically that the film resolution should be eight times 5 l/mm or 40 l/mm. The reciprocal of this, 0.025 mm, gives the spacing between lines, and the diameter of a circle at the limit of human discrimination. This circle is called the "circle of confusion."

The circle of confusion thus depends on the degree of magnification of the image with respect to the size of a print taken with a standard lens. As may be seen throughout these notes, the diameter of this circle, c, is an important variable. For DOF, setting it properly means that those objects at the DOF limits will just be resolvable in a print. For a 35 mm camera, it should be at most about 0.025/mm, and should decrease as the print size increases past 8x10.

6.5 DOF and format

There is always heated argument about the DOF achieved from various formats. The general view is that larger formats have smaller DOF. To a large extent this opinion derives from a selective modification of the parameters: leaving some fixed and changing others.

If one fixes the f-number, then the DOF decreases as the format increases, but this is not so if the f-number is scaled as the other parameters are scaled. Figure (10) on the left shows the DOF when $N = 16$, $c = 0.025$ mm, $m = 0.01$, and $f = 50$ mm. The x axis indicates the format in terms of the size of the short side of the film – 1 for 35 mm, 4 for 4x5. The lens to subject distance, u, is approximately constant at 5 m for the x's shown. One concludes from this that DOF decreases with format size when the f-number is fixed, but remains unchanged or slightly increases when the f-number -number is scaled with the other parameters.

For macro photography, with $m = 1$ and the other parameters as above, Figure (10) on the right shows the DOF in the fixed and scaled cases – note that the DOF scales are in millimeters on this graph. One has $u = 100$ mm for the 35 mm format and $u = 400$ mm for the 4x5 format.

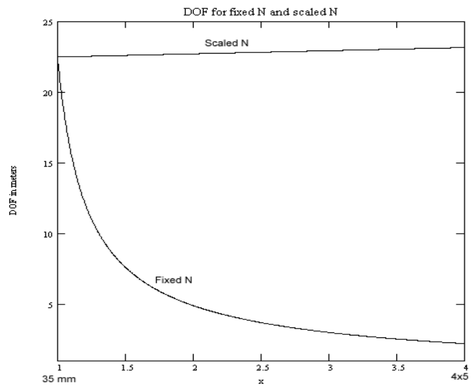
6.6 The DOF equation

An interesting equation similar to the Gaussian lens equation may be derived for the near and far DOF limits.

From figures 8 and 9 one sees

$$\frac{d}{c} = \frac{v+x}{x} = \frac{v-y}{y} = r,$$

[3] One who's diameter is approximately equal to its focal length



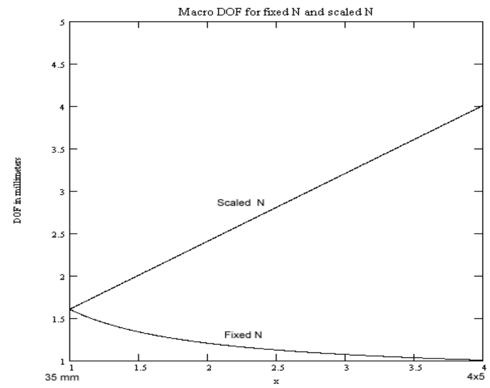


Figure 10: DOF for Fixed and scaled N

thus

$$\begin{aligned} \frac{1}{v+x} &= \frac{1}{v}\frac{r-1}{r}, \\ \frac{1}{v-y} &= \frac{1}{v}\frac{r+1}{r}, \end{aligned}$$

and since

$$\begin{aligned} \frac{1}{Z_n} + \frac{1}{v+x} &= \frac{1}{u} + \frac{1}{v}, \\ \frac{1}{Z_f} + \frac{1}{v-y} &= \frac{1}{u} + \frac{1}{v}, \end{aligned}$$

one has by addition

$$\frac{1}{Z_n} + \frac{1}{Z_f} = \frac{2}{u} + \frac{2}{v} - \frac{1}{v}\left(\frac{r-1}{r} + \frac{r+1}{r}\right),$$

or the *DOF equation*

$$\frac{1}{Z_n} + \frac{1}{Z_f} = \frac{2}{u}. \tag{25}$$

A more straightforward derivation is to sum the reciprocals of equations (14) and (17).

6.7 Hyperfocal distance

The hyperfocal distance in equation (15) is the distance at which the far limit of DOF becomes infinite. Substituting this as u into the formula for the near limit, equation (14) gives

$$Z_n = \frac{H}{2},$$

thus focusing at the hyperfocal distance puts everything in focus from half this distance to infinity. It is worth noting that $H = \frac{f^2}{Nc} + f = f(1 + \frac{f}{Nc})$.

Another often quoted definition for "hyperfocal distance" is the value of Z_n when $u = \infty$. As u increases, Z_n in equation (14) may be seen to approach $f^2/(Nc)$, a good approximation to H.

As a practical matter, the hyperfocal focus is best determined from distances on the image side of the lens. Simple substitution into equation (15) shows

$$v_H = f + Nc = f(1 + \frac{Nc}{f}), \tag{26}$$

where v_H is the position of the standard when focused on the hyperfocal distance. Thus one may focus on infinity, and then move the standard by Nc to obtain focus at the hyperfocal distance.

6.8 Approximations

These equations simplify by noting that in many cases $u - f \approx u$, giving the usually cited formulas

$$H' \approx \frac{f^2}{Nc}. \tag{27}$$

$$Z'_n = \frac{uf^2}{f^2 + uNc} = \frac{Hu}{H + u}, \tag{28}$$

$$Z'_f = \begin{cases} \frac{uf^2}{f^2 - uNc} = \frac{H'u}{H' - u} & \text{for } u < H' \\ \infty & \text{for } u \geq H' \end{cases} \tag{29}$$

$$DOF' = \begin{cases} \frac{2Ncu^2f^2}{f^4 - (Ncu)^2} = \frac{2H'u^2}{H'^2 - u^2} & \text{for } u < H' \\ \infty & \text{for } u \geq H' \end{cases} \tag{30}$$

The approximate DOF is slightly larger than the exact DOF. A calculation shows that $Z_n(1 - Nc/f) \approx Z'_n$, and $Z_f(1 - Nc/f) \approx Z'_f$, and that $DOF' \approx DOF + (Z_n + Z_f)Nc/f$. It is hard to imagine situations in which these small differences will matter.

6.9 Focus given near and far limits

6.9.1 Object distances

An interesting question, is "what is the focus distance, given the near and far DOF limits?"

The DOF Equation, equation(25), is

$$\frac{1}{Z_n} + \frac{1}{Z_f} = \frac{2}{u},$$

which when solved for u gives

$$u = \frac{2Z_n Z_f}{Z_n + Z_f} = \frac{2\alpha x}{1+\alpha}, \tag{31}$$

where $x = Z_n$ and $\alpha = Z_f/Z_n$. Note also that u is the product of the limits divided by their average, and that the commonly stated one third rule[4] is correct, non-trivially, only when $\alpha = 2$; although many will be satisfied when using it in the range of, say, $2 \pm \frac{1}{2}$.

One may solve for N given f in equation (14), obtaining

$$N = \frac{f^2}{c} \frac{Z_f - Z_n}{(2Z_f Z_n - f(Z_f + Z_n))} = \frac{f^2}{c} \frac{\alpha - 1}{(\alpha(2x - f) - f)}, \tag{32}$$

and for f given N:

$$\begin{aligned} f &= \frac{1}{2} Nc \frac{Z_f + Z_n}{Z_f - Z_n} \left(-1 + \sqrt{1 + \frac{8Z_f Z_n}{Nc} \frac{Z_f - Z_n}{(Z_f + Z_n)^2}} \right), \\ &= \frac{1}{2} Nc \frac{\alpha + 1}{\alpha - 1} \left(-1 + \sqrt{1 + \frac{8\alpha x}{Nc} \frac{\alpha - 1}{(\alpha + 1)^2}} \right). \end{aligned}$$

Thus from two distances x and αx, with $\alpha > 1$ one may obtain the focus distance and either the f-number or the focal length that will make these two distances the near and far DOF limits. It is useful to note that equation (31) ranges between $1x$ and $2x$, and that $2x$, for $x > 0$, corresponds to focus at the hyperfocal value. It is also interesting that approximations for f and N may be obtained by equating the focus distance to the hyperfocal distance.

6.9.2 Image distances

The expressions simplify considerably when one uses the image distances z_n and z_f corresponding to Z_n and Z_f respectively. For u one has

$$\frac{1}{u} = \frac{1}{f} - \frac{1}{\theta}, \tag{33}$$

where θ is the harmonic mean of z_n and z_f:

$$\theta = \frac{2z_n z_f}{z_n + z_f}.$$

Equation (33) may also be obtained from equation (2).

[4] "The focus point is one third of the distance between the near and far limits."

Since $\theta = \lambda/\mu$, where λ is the square of the geometric mean and μ the arithmetic mean of z_n and z_f, and since all three means differ by at most about 1%, one can write equation (33) as

$$\frac{1}{u} \approx \frac{1}{f} - \frac{1}{\mu}.$$

For N given f one may write from equation (32)

$$cN = \rho\frac{f}{\mu} = \frac{\rho}{1+m}, \tag{34}$$

where $\rho = (z_n - z_f)/2$, and m is magnification.

When the magnification is small, one has the approximation

$$cN \approx \rho,$$

which is likely that used by Sinar for their DOF calculation.

The expression for f given N is of course

$$f = \frac{\mu c N}{\rho}.$$

7 Depth of field, depth of focus

The function relating DOF and ρ may be obtained by assuming the depth of focus range 2ρ is equally divided by the focus point v. Thus $(v-\rho)$ and $(v+\rho)$ are the conjugates of Z_f and Z_n. This gives

$$DOF = Z_f - Z_n = \frac{2f^2\rho}{(v-f)^2 - \rho^2} = \frac{2f^2\rho}{(mf)^2 - \rho^2}, \tag{35}$$

from which one may express ρ as a function of DOF and m:

$$\rho = \frac{f^2}{DOF}\left(-1 + \sqrt{1 + \left(\frac{mDOF}{f}\right)^2}\right). \tag{36}$$

Equation (34) may be used to find $m = (\rho - Nc)/Nc$, which then gives ρ as a function of DOF:

$$\rho = \frac{Ncf^2}{f^2 - (Nc)^2}\left(\left(1 + \frac{Nc}{DOF}\right) + \sqrt{\left(1 + \frac{Nc}{DOF}\right)^2 - \left(1 - \left(\frac{Nc}{f}\right)^2\right)}\right),$$

or

$$\rho = \frac{Ncf^2}{f^2 - (Nc)^2}\left(\left(1 + \frac{Nc}{DOF}\right) + \sqrt{2\frac{Nc}{DOF} + \left(\frac{Nc}{DOF}\right)^2 + \left(\frac{Nc}{f}\right)^2}\right).$$

8 Fuzzy images

Points image as circles on the film. Some of the image circles are so small that the eye cannot distinguish them on a print or slide. These are those between the DOF limits. Points in other regions, image as circles which the eye can distinguish, and objects made up of such circles are fuzzy. Knowing the degree of fuzziness for an object can be useful; as when one wants letters on a sign in the background to be unreadable. This may be done by calculating the size of an object at a given distance, which will produce an image with diameter k times the diameter of a circle corresponding to a point image at that distance. Thus an object with $k = 2$ will be quite fuzzy and impossible to recognize, while one with $k = 10$ will probably be recognizable, although still fuzzy.

Figure 11 shows the three planes as well as a plane, P, and its image plane, IP with a circular[5] object of diameter S on P. The image of this object has diameter r on IP, and t on FP. The diameter of the aperture is d The diameter of the image of an object point is shown as p. If P were the near DOF plane, then p would equal c. The problem is to calculate S corresponding to a multiple, say k, of p.

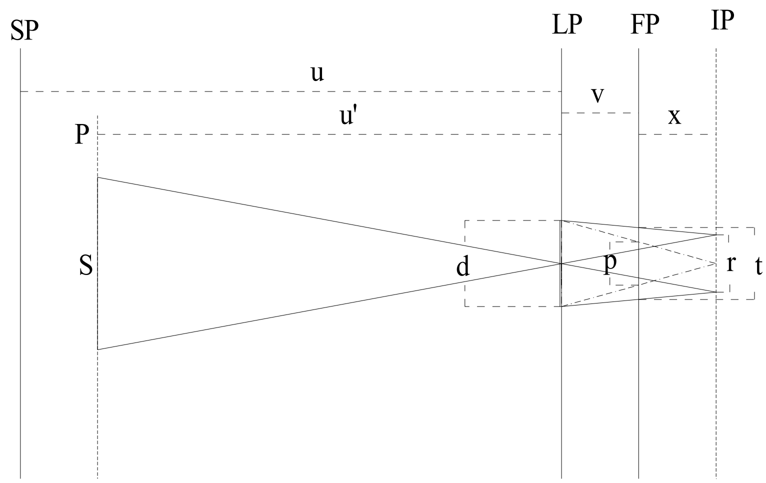


Figure 11: Object in front of SP

The magnification of the object with respect to IP is $r/S = (v + x)/u'$, thus

$$r = S\frac{v + x}{u'} = S\frac{f}{u' - f},$$

from equation (3).

From similar triangles $(d - r)/(t - r) = (v + x)/x$ one finds

$$t = \frac{rv + xd}{v + x}.$$

[5] A flat disk without depth.

Again using equation (3), one finds

$$x = v + x - v = \frac{u'f}{u'-f} - \frac{uf}{u-f} = \frac{f^2(u-u')}{(u'-f)(u-f)}. \tag{37}$$

From similar triangles $p/d = x/(v+x)$, and one obtains

$$p = \frac{xd}{v+x}.$$

Let t be a multiple of p, say $t = kp$, then

$$t = \frac{rv+xd}{v+x} = kp = k\frac{xd}{v+x},$$

thus

$$(k-1)xd = rv = r\frac{uf}{u-f} = S\frac{f}{u'-f}\frac{uf}{u-f},$$

and using equation (37),

$$S = (k-1)d\frac{(u'-f)(u-f)}{f^2u}x = (k-1)\frac{f}{N}\frac{u-u'}{u} = (k-1)\frac{xf^2}{Nv(v+x-f)}, \tag{38}$$

with $N = f/d$.

At the near DOF limit, this becomes

$$S = (k-1)\frac{f}{N}\left(1 - \frac{Z_n}{u}\right).$$

It is worth noting that the above derivation holds if S is not symmetrical about the lens axis.

Figure 12 shows the planes with P to the left of SP. The argument is essentially the same as above. The only small problem is finding the similar triangles involving t. The inset shows the triangles, which give

$$\frac{d+t}{t-r} = \frac{v}{y},$$

thus,

$$t = \frac{rv+dy}{v-y}.$$

The equation is

$$S = (k-1)\frac{f}{N}\frac{u'-u}{u} = (k-1)\frac{yf^2}{Nv(f+y-v)}, \tag{39}$$

and when P is the far DOF limit:

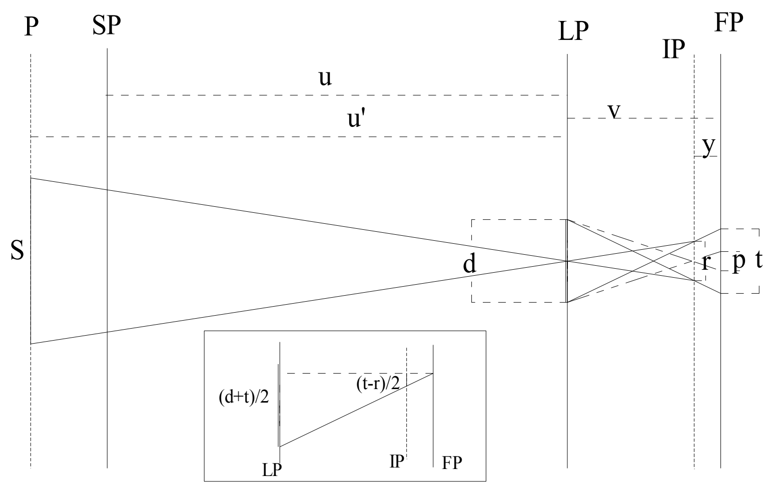


Figure 12: Object behind SP

$$S = (k-1)\frac{f}{N}\left(\frac{Z_f}{u} - 1\right).$$

The derivation does not depend on the symmetry of S about the lens axis. Useful expressions for t and S in terms of each other are:

$$\begin{aligned} t &= \frac{f|u-u'|d}{u'(u-f)} + \frac{fu}{u'(u-f)}S, \\ S &= \frac{u'(u-f)}{fu}t - \frac{|u-u'|d}{u}. \end{aligned}$$

9 Effects of diffraction on DOF

9.1 Theory

Light from a point source, diffracted by a circular opening, is focused by a lens not as a geometrical point, but as a disk of finite radius surrounded by dark and bright rings. This is due to the fact that the path of light near an edge is longer than that away from the edge causing a phase difference, and complete interference occurs at a certain distance from the center, and at harmonic distances further away.

From a study of the appearance of the diffraction patterns of closely spaced points, Lord Rayleigh concluded that two equally bright point sources could just be resolved by an optical system if the central maximum of the diffraction pattern of one source coincided with the first minimum of the other. The disk defined by this first minimum is called an Airy disk. The angle α from the center of the lens subtended by two points with this spacing is given by $sin(\alpha) = (1.22\lambda)/(nD)$, where λ is the wavelength of light, D is the diameter of the aperture, and n the index of refraction. For an object at distance u, one finds the object distance to be $s = sin(\alpha)u$. Since $u/D = f(1+\frac{1}{m})/D = N(1+\frac{1}{m})$, where f is focal length, m is magnification, and N is the f-number, one has:

$$s = 1.22N(1+\frac{1}{m})\lambda/n \approx \frac{N(1+\frac{1}{m})}{K_\lambda n}\text{mm}, \tag{40}$$

Since visible light ranges from about 400×10^-6 mm to about 700×10^-6 mm, K_λ varies from about 1000 $\frac{1}{(mm)}$ to about 2000 $\frac{1}{(mm)}$. Often one takes $K_\lambda = 1500 \frac{1}{(mm)}$ for a medium wavelength $\lambda = 550 \times 10^-6$ mm: see Sears (1958), p260.

The distance s' on the other side of the lens is given by $s' = ms(n/n')$, where n' is the index of refraction on the image side of the lens. This is

$$s' \approx \frac{N(1+m)}{K_\lambda n'}\text{mm}, \tag{41}$$

The spacial frequency $1/s$ gives lines per millimeter, l/mm, as the frequency of two just separable lines. The diameter of the diffraction circle of confusion, c, is s. Test charts allow the selection of "just separable" pairs of lines, called either "line pairs per millimeter", lp/mm, or simply "lines per millimeter", l/mm. These correspond to the spacial frequency $1/s$. For test charts, black bars are used separated by spaces of the same width, thus a spacial cycle consists of a black bar and a space, with $s/2$ the common width.

It is generally agreed that normal human vision can just resolve about 5 l/mm on a test chart at the minimum distance of distinct vision (about 250mm). This may be confirmed by inspecting an Edmund USAF pattern. It follows that for human vision, $s \approx 1/5 = 0.2$ mm or 0.008 in. Assuming the human eyeball is about 25 mm, this gives $m = 0.1$ which when substituted in equation (40) gives $N \approx 25$ mm. Since $N = f/D$, and $f \approx 25$ mm, this gives $D \approx 1$mm, which is a bit small, but in the ball park (In bright light the iris can close down to about 2 mm.).

9.2 Data

Diffraction affects resolution so that the parameter c used in equations, such as equation (34), changes as other parameters change. Table 1 [6] shows resolution in l/mm as a function of N and depth of focus, δ mm [7], about the point of focus. The 35 observations in this table were obtained by using a 150 mm Sironar lens with Velvia. The contrast between the dark lines and the white background was approximately 5 stops, placing the contrast ratio, 35, approximately midway

[6] records 153-157

[7] Note: $\delta = 2\rho$ in equation (34)

	δ mm				
f-number	0	2.5	5	7.5	10
64	23	23	20	20	20
45	23	23	26	23	23
32	26	32	23	16	10
22	32	26	14	9	6.4
16	32	18	9	6.4	4.5
11	23	18	6.4	4.5	3.2
8	26	9	4.5	2.5	2.25

Table 1: Raw resolution data for Velvia

	δ mm				
f-number	0	2.5	5	7.5	10
64	22	20	20	20	17
45					
32	28	25	20	17	11
22	28	22	14	8	9
16					
11	31	11	6.1	3.9	3.1
8	28	4.3	3.1	2.2	1.7

Table 2: Raw resolution data for TMax 100

between the usual 1.6 and 100 contrast ratios. The USAF target was 1500 mm distant from the lens giving a magnification of 1/9.

The data was smoothed by a low order polynomial and is shown as Figure 13. The residuals had a sample standard deviation of 3.1 l/mm; which is of the order of experimental error, since one can read the USAF charts only to the nearest 2 l/mm. Table 2 [8] shows a set of data taken with TMAX 100 using a 210mm Sironar lens – all other parameters are the same as for Velvia.

In Section 9.6 it will be seen that the data for $\delta = 0$ does not agree with equation (43). This suggests that the data may be in error. It seems likely that the focus when taking the data was not at $\delta = 0$, but at some small value, perhaps $\delta = 0.5$ or larger.

To preserve perspective, one should view a picture at a distance consistent with that of the taking lens. For normal lens's where the focal length is approximately equal to the film size, this means that one views a picture x inches high from x inches. For a 4x5 film, this means that an 8x10 will be at the distance of distinct vision, and so the diameter of the on-film circle of confusion should be one half s or 0.1 mm, corresponding to a spacial frequency of 10 l/mm.

The locus of equation (34) is marked on figure 13 by the line labeled $c = 0.1$. Lines for $c = 0.05$ and $c = 0.2$ are also shown. The $c = 0.1$ lines seems to be

[8] record 149

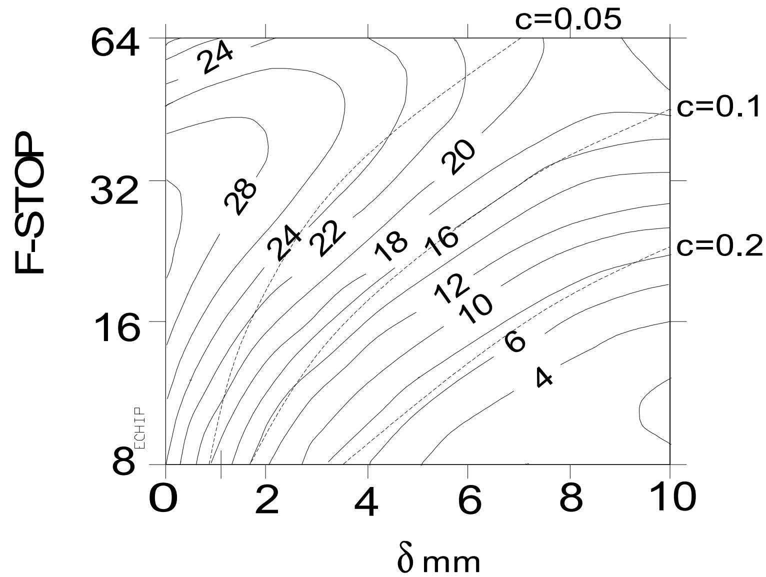


Figure 13: Resolution as a function of N and δ for m=1/9

somewhat above the 10 l/mm contour, but the data has a standard deviation of at least 2 l/mm. Since 4x5 prints are often printed at 16x20, one would need a film resolution of 20 l/mm, which corresponds to $c = 0.05$. The line $c = 0.05$ shown in figure 13 is slightly above the 20 l/mm contour. Both the data and the choice of fitting model could be improved upon. The results are good enough, however, to show that equation (34) is reasonable.

As an additional check, an array of dollar bills was photographed and the film examined under magnification. In figure 17 the dollars were arranged 30 mm apart on a slanting board. A Pentax 645 camera with a 150 mm lens at f32 was positioned 1500 mm from the central pair of bills (numbers 8 and 9, marked with dots on their corners). The DOF using $c = 0.05$ is 290 mm, with the front and back DOF being 130 mm and 160 mm. This ranges from bill number 5 through bill number 12. The magnification is $1/9 = 150/(1500 - 150)$, hence a 9 power loupe will bring the image to full size. An 8 power loupe was used. Noticeable degredation seems to occur in the neighborhood of bills 5 and 12, so the choice of $c = 0.05$ for a 645 format seems reasonable.

9.3 Resolution

It is useful to observe how the contours change in figure 13. For large δ, the lines per millimeter increase steadily with N because of the shrinking aperture. For small δ's, however, N increases to a maximum and then decreases. This is due to the fact that lines per millimeter, increases directly with N for depth of focus, and inversely for diffraction. Figure 15 indicates the effect of decreasing

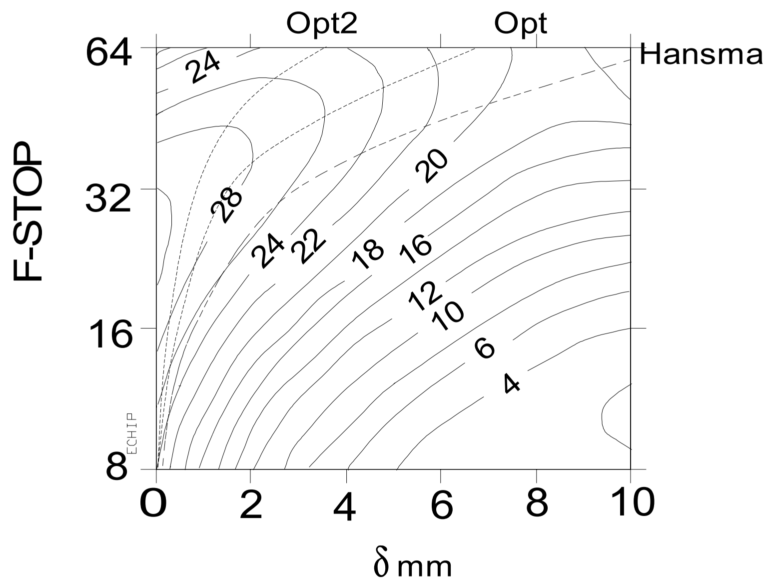


Figure 14: Optimum N as a function of δ for m=1/9

the aperture on the image circle. This figure may be compared with figures 8 and 9. For a fixed depth of focus with a constant distance between IP and FP, decreasing the aperture is seen to decrease the image circle, thus increasing the resolution. The effect of diffraction on resolution is just the opposite, since decreasing the aperture makes the image more diffuse and decreases the resolution.

For depth of focus, equation (34) gives

$$L_\delta = (2N(1+m))/\delta.$$

From equation (41)[9] for diffraction, one has

$$L_d = K_\lambda/[N(1+m))].$$

The point at which the diameter of the depth of focus circle of confusion equals the diameter of the diffraction circle of confusion is obtained by equating L_δ to L_d, giving

$$N = \frac{\sqrt{K_\lambda \delta/2}}{1+m}, \tag{42}$$

which is the maximum N for a given δ, since the product $L_d L_\delta = 2K_\lambda/\delta$ does not depend on N.

[9]Equation (40) is not appropriate since here one needs to compare resolutions on the image.

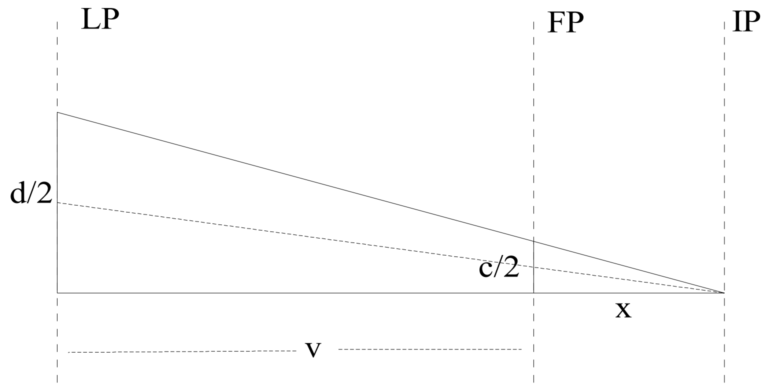


Figure 15: Effect of aperture on the image circle.

Hansma (1996) attempted to combine depth of focus and diffraction. He chose an empirical combination of depth of focus and diffraction by taking the RMS of the two. He seems to have made several errors such as using $2s$ for the diffraction circle of confusion, ignoring the magnification factor in equation (40), and even confusing equations (40) and (41). In any case, Hansma found the optimum N to be $N = \sqrt(375\delta)$ for $m = 0$. This is $1/\sqrt(2)$ of equation (42). The particular choice of RMS has no effect, since any linear combination of powers of the two circle of confusion expressions will produce the same optimum.

Equation (42) seems to agree better with the observed optimum than does Hansma's. Equation (42) is marked "Opt" in figure (14). The curve "Opt2" represents $2L_\delta = L_d$, which may help to visualize the effect of diffraction on resolution. The apparent intersection of the contours with the $\delta = 0$ vertical are likely to be an artifact the polynomial form and measurement error.

9.4 Format considerations

It is worth noting that equation (42) does not depend on format. Although the resolution values will differ among formats, the relationship between N and δ does not change. Equation (36) gives $\rho = \delta/2$ as a function of magnification, m, DOF, and f. The limit of equation (36) as DOF goes to infinity is $\delta = 2mf$. This assumes depth of focus on both sides of the focus. When the far limit is infinite, the near limit is half the hyperfocal distance, and this works out to $\delta \approx 2mf$ where m is the magnification for an object at half the hyperfocal distance. For a fixed object size at a fixed distance from the lens, the m and f for 35 mm are both about 1/3 that for 4x5, reducing mf by about 1/9. It is important to note that these changes do not effect DOF, as may be seen from equation (35) by substituting $f/3$, $m/3$ and $\rho/9$.

Figure (16) shows equation (42) for this 35 mm scaling.

The diffraction resolution L_d is affected in a very interesting way by format. From Section 6.5, N must change as format changes in order to keep DOF

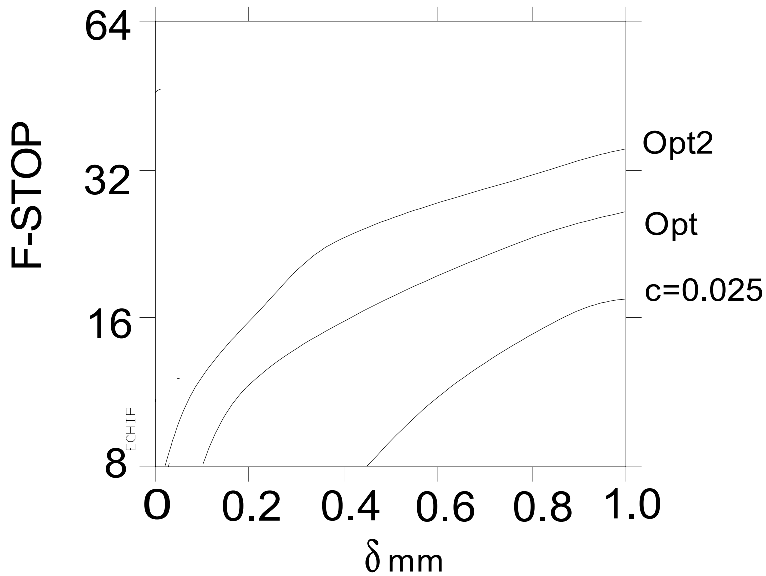


Figure 16: Optimum N as a function of δ for 35 mm

constant: but L_d is inversely proportional to $N(1+m)$, hence smaller formats must have greater resolution. The change from 4x5 to 35 mm is about 1/3, hence 35 mm images must produce about 3 times the resolution of 4x5 images, and they do!

Another way to view this is to examine equation (41) with $N = f/A$, where A is the aperture. For any format, the smallest aperture that is practical is about 2 mm, hence resolution is a function of the distance between lens and film ($f(1+m)$). Since this is smaller for smaller formats, their resolutions must be greater.

9.5 Minimum aperture

The curve from equation (34) using the appropriate c lies well below the Opt curve in figures (14) and (16). The δ at which the Opt curve meets the c curve is $2K_\lambda \times c^2$, which for $K_\lambda = 1500$, is about 1.8 for 35 mm and 30 for 4x5 – well outside any practical range. For any format the point (N, δ) is usually well away from the optimum curve, except at extreme depth of focus limits for the format. At such extremes the point (N, δ) comes close to the optimum, and any substantial increase in N will decrease resolution. Lens manufacturers use this to choose the minimum f-number for their lenses.

Figure 17: DOF determination

9.6 Theoretical curves

One must combine L_d and L_δ in order to find the actual resolution. The combination should approach one of the resolutions as the other becomes extreme. One choice is $(1/L_\alpha)^\alpha = (1/L_\delta)^\alpha + (1/L_d)^\alpha$, which gives

$$L_\alpha = \frac{K_\lambda N(1+m)}{((K_\lambda \delta/2)^\alpha + (N(1+m))^{2\alpha})^{1/\alpha}}, \tag{43}$$

Substituting equation (42) into this gives the maximum $\overline{L}_\alpha = 2^{-1/\alpha}\sqrt{\frac{K_\lambda}{\delta/2}}$. The agreement between equation (43) and the data is good, suggesting that it captures a fundamental aspect of the problem. Values calculated from $\overline{L}_\alpha$ agree reasonably well with figure (14) in view of the 2 l/mm error in the data, except for $\delta = 0$. Table 3 shows values calculated from equation (43), which may be compared with tables 1 and 2. A contour plot of this data is shown in figure (18), and may be compared with figure (13). The data at $\delta = 0$ was not used to make the contour plot, thus the plot does not show the singularity at $\delta = 0$.

The predictions for the 35 mm case produce resolutions of a magnitude similar to what one expects from 35 mm. The implication of this is that differences in the formats are principally the result of size and not of superior lens quality for smaller formats! Both lens quality and focus precision play a role, but the higher resolutions for small format appear to be due to operating in the range of smaller δ's. This is consistent with the conclusion in Section 9.4.

	δ mm						
f-number	0	1	2	4	6	8	10
64	21.1	20.9	20.2	19.4	15.8	13.6	11.8
45	30.0	28.7	25.7	19.2	14,6	11.5	9.5
32	42.2	36.3	27.2	16.4	11.4	8.7	7.0
22	61.4	38.2	22.7	12.0	8.1	6.1	4.9
16	84.4	32.8	17.4	8.8	5.9	4.4	3.6
11	122.7	24.0	12.2	6.1	4.1	3.2	2.4
8	168.8	17.7	8.9	4.4	3.0	2.2	1.8

Table 3: Theoretical resolution from eq (43) with $\alpha = 2$.

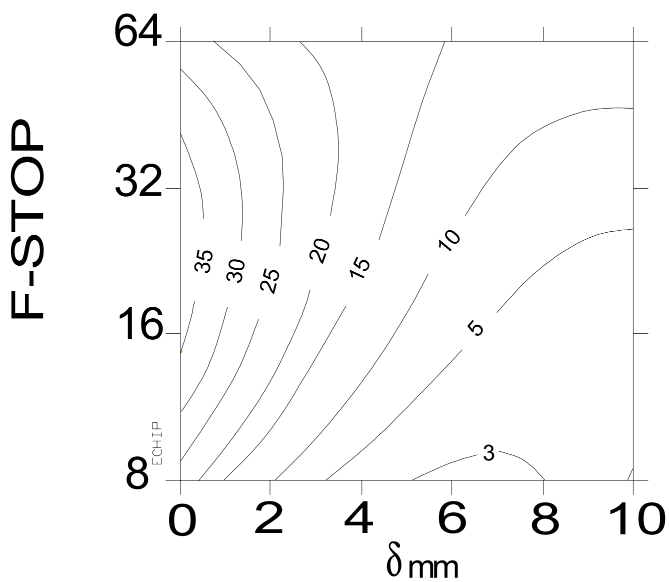


Figure 18: Contours for theoretical resolution from eq (43) with $\alpha = 2$.

10 Depth of field for a tilted lens

10.1 Near and far DOF equations

The DOF equations above hold for tilted lenses with minor modifications. Figure 19 shows the three planes SP, LP, and FP with a ray at angle ρ. Planes F and I are parallel to LP and have been drawn through the intersections of the ray with SP and FP. The distances of F and I from LP are u and v, respectively. Equation (14) may thus be used to find the near DOF which is labeled u' in figure 19. Let h be the distance from the lens center L along the ray to SP, then $u = h\cos(\phi+\rho)$, thus equation (14) becomes

$$u' = \frac{hf^2\cos(\phi+\rho)}{f^2 + [h\cos(\phi+\rho) - f]Nc},$$

and the near DOF along the ray is $Z_n = u'/\cos(\phi+\rho)$.

There is one minor problem: the diameter of the circle of confusion is not correct. The above assumes the image is on I not FP. The image on FP is an ellipse, and its maximum diameter varies slightly with ρ; thus the near DOF for a ray at angle ρ is:

$$Z_n(\rho) = \frac{hf^2}{f^2 + [h\cos(\phi+\rho) - f]g}. \tag{44}$$

where $g \approx Nc$.

The graph of this equation, with constant g, is a straight line, shown as NP in figure 19. It corresponds to the image plane IP behind FP, and passes through the pivot point.

Similar calculations follow through when the ray is below the lens axis, and following the sign convention used above, result in equation (44) with ρ a negative quantity.

The far DOF limit is derived in the same way and is

$$Z_f(\rho) \approx \begin{cases} \frac{hf^2}{f^2 - [h\cos(\phi+\rho) - f]g} & \text{for } h < H(\rho), \\ \infty & \text{for } h \geq H(\rho) \end{cases} \tag{45}$$

where $H(\rho) = f^2/(g\cos(\phi+\rho))$.

10.2 Near and far DOF equations in terms of ρ

It is often more useful to calculate these limits in terms of ρ instead of h. To this end, it may be seen from figure (20) that

$$\frac{h\cos\rho}{\tan\gamma} = J + h\sin\rho,$$

hence

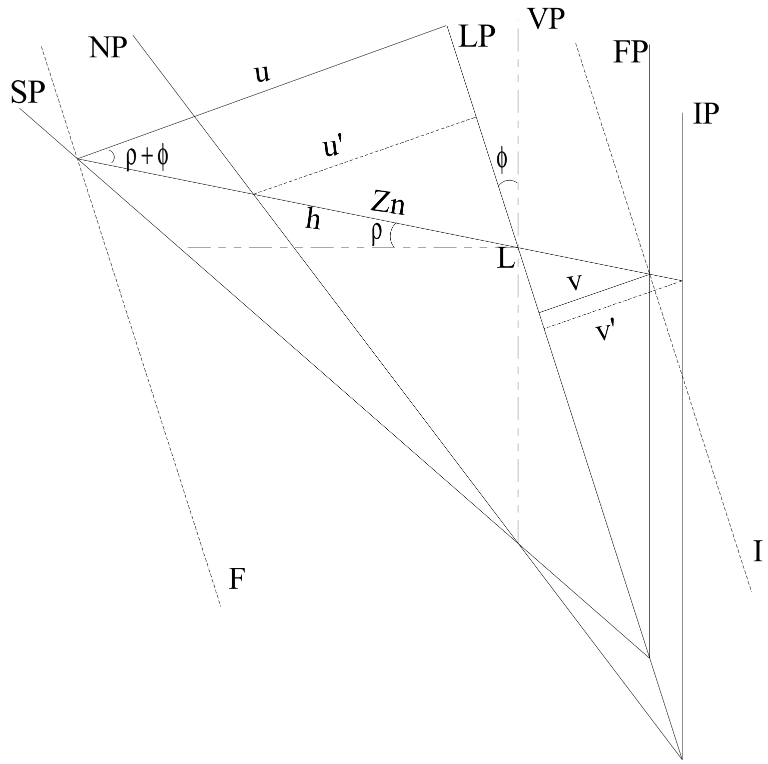


Figure 19: Tilted DOF diagram

$$h = \frac{J \sin\gamma}{\cos(\rho + \gamma)},$$

and thus, using $J = f / \sin\phi$,

$$Z_n(\rho) = \frac{f^2 \sin\gamma}{f \sin\phi \cos(\rho + \gamma) + g \cos\rho \sin(\gamma + \phi)}, \tag{46}$$

and

$$Z_f(\rho) = \begin{cases} \frac{f^2 \sin\gamma}{f \sin\phi \cos(\rho+\gamma) - g \cos\rho \sin(\gamma+\phi)} & \text{for non-negative denominator} \\ \infty & \text{for negative denominator} \end{cases} \tag{47}$$

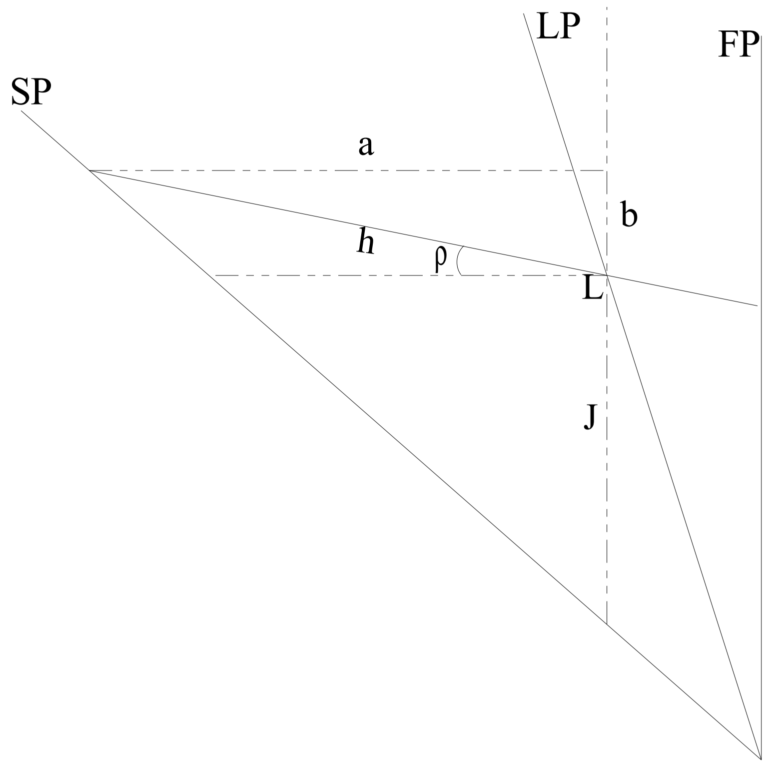


Figure 20: Finding h from ρ

10.3 Focus given near and far limits

10.3.1 Object distances

Just as for parallel planes, one may obtain the focus distance from the near and far limits. An equation similar to the DOF Equation, equation (25), holds for $Z_n(\rho)$ and $Z_f(\rho)$;

$$\frac{1}{Z_n(\rho)} + \frac{1}{Z_f(\rho)} = \frac{2}{h}, \tag{48}$$

which is most easily seen by assuming g constant and summing the reciprocals of equations (44) and (45).

The DOF for a given ray angle ρ is

$$DOF(\rho) = \frac{2g(h\cos(\phi+\rho) - f)hf^2}{f^4 - [g(h\cos(\phi+\rho) - f]^2}.$$

An expression for N given f is

$$N = \frac{f^2}{c}\frac{Z_f(\rho) - Z_n(\rho)}{2Z_n(\rho)Z_f(\rho)\cos(\phi+\rho) - f(Z_n(\rho) + Z_f(\rho))}, \tag{49}$$

and for f given N:

$$f = \sqrt{g\frac{2Z_n(\rho)Z_f(\rho)\cos(\phi+\rho) - fZ_n(\rho) + Z_f(\rho))}{Z_f(\rho) - Z_n(\rho)}}.$$

One might use these expressions in calculating the lens tilt by replacing the distance to SP along one of the rays with the distances of the desired near and far limits.

10.3.2 Image distances

Using equation (2) in the form $hf/(h\cos(\phi+\rho) - f)$ in equation (49) gives

$$cN = \delta\frac{f}{\mu},$$

where $\delta = (z_n(\rho) - z_f(\rho))/2$ and $\mu = (z_n(\rho) + z_f(\rho))/2$, with $z_n(\rho)$ and $z_f(\rho)$ the conjugates of $Z_n(\rho)$ and $Z_f(\rho)$ via equation (2). Since the parameters c, N, and f are constant in this formula, the ratio δ/μ is also constant for all ρ, and one may choose to measure δ and μ along the original lens axis, with $\rho = 0$.

In this case, one has $\mu\cos(\phi) = f(1+m)$, and remembering that m is a constant for all angles, one has

$$cN = \frac{\delta\cos(\phi)}{1+m}. \tag{50}$$

This may be compared with equation (32) in section 6.9.2, which is equivalent for practical purposes, since ϕ is never very large.

The expression for f given N is of course

$$f = \frac{\mu c N}{2\delta\cos(\phi)}.$$

10.4 The graph of the DOF limits

The tangent of the graph of $Z_n(\rho)$ is

$$T = \tan\rho + \frac{J}{Z_n(\rho)\cos\rho}.$$

On substituting $Z_n(\rho)$ as given by equation (46) into this, and using $J = f/\sin\phi$, one has

$$T = \frac{1}{\tan\gamma} + \frac{g}{f}\left(\frac{1}{\tan\phi} + \frac{1}{\tan\gamma}\right),$$

which shows that the graph depends on ρ only through g, and hence if g is constant, the graph is a straight line. Similar comments apply to $Z_f(\rho)$: the equation is the same, save for a negative sign in front of g.

An interesting observation is that the rise of the $Z_n(\rho)$ graph above SP at the hyperfocal distance is approximately equal to J. Let $H = f^2/g$, and note that $H/\tan\gamma$ is on SP, then one sees from T, above that the rise above SP is

$$J\cos\phi + \frac{f}{\tan\gamma}.$$

This is practically different from J, only for very small γ or large ϕ. The fall for $Z_f(\rho)$ is also approximately equal to J.

Since $\arctan(T)$ is $\frac{\pi}{2} - \gamma_+$, where γ_+ is the angle of the near DOF plane with respect to the vertical, one has

$$\begin{aligned} \tfrac{1}{\tan(\gamma_+)} &= \tfrac{1}{\tan\gamma} + \tfrac{g}{f}\left(\tfrac{1}{\tan\phi} + \tfrac{1}{\tan\gamma}\right), \text{ and} \\ \tfrac{1}{\tan(\gamma_-)} &= \tfrac{1}{\tan\gamma} - \tfrac{g}{f}\left(\tfrac{1}{\tan\phi} + \tfrac{1}{\tan\gamma}\right), \end{aligned}$$

with γ_- the angle of the far DOF plane with respect to the vertical.

10.5 Variation of maximum diameter of the ellipse of confusion with ρ

We will show that the variation of the maximum diameter of the ellipse of confusion is small, and consequently the DOF limits are approximately linear.

Figure 21 is a modified portion of figure 19. The aperture is the slightly darkened portion of LP with diameter d. The corresponding image on I, centered[10] about the slanted ray, has diameter[11] δ. On FP, the image is an ellipse with extents $c1$ above and $c2$ below the ray. We desire $c = c1 + c2$ to be constant, and thus must find δ.

The distance, j, along the ray from LP to FP is $v/\cos(\rho+\phi)$, and similar triangles give $\delta/x = d/(j+x)$, or

$$x = \frac{\delta v}{(d-\delta)\cos(\rho+\phi)},$$

or after some manipulation:

$$x = \frac{\delta}{d-\delta}\frac{f\sin\gamma}{\cos\rho\sin(\gamma-\phi)}.$$

Let $p = \frac{\delta}{2}\cos\phi$, then one may write

[10] LP and I are parallel, hence the ray divides δ into two equal parts. Even more, the size of δ is constant because $x/j = \delta/(d-\delta)$ is constant – circles of confusion remain circles and do not turn into ellipses.

[11] Note: $g = N\delta$.

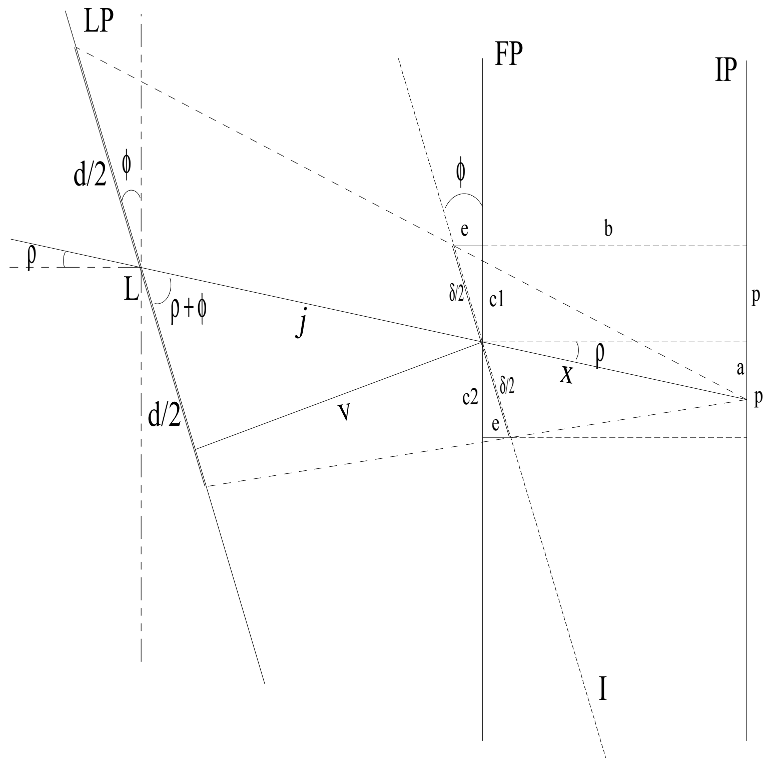


Figure 21: Ellipse of confusion diameter for tilted lenses

$$c1 = p - e\frac{p+a}{b+e},$$
$$c2 = p + e\frac{p-a}{b-e},$$

where $e = \frac{\delta}{2}\sin\phi$, $a = x\sin\rho$, and $b = x\cos\rho$.

Adding and simplifying gives

$$c = \delta\cos\phi + \delta\sin\phi\left\{\frac{(\frac{\delta}{2})^2\cos\phi\sin\phi - x^2\sin\rho\cos\rho}{(x\cos\rho)^2 - (\frac{\delta}{2}\sin\rho)^2}\right\} \tag{51}$$

and for paraxial rays, the expression on the right differs negligibly from δ; hence one may reasonably take $g = Nc$ as a constant in equations (46) and (47). Figure 22 shows a comparison of the near DOF plane, with a graph of equation (46) using $g = N\delta$ obtained by solving equation (51) for δ as a function of c. The parameters are $\phi = 5°$, $\gamma = 80°$, $F = 150$mm, and $N = 32$. The dotted curve, representing the graph of equation (46), becomes infinite when $\rho \approx 22.6°$: the angle of the near DOF plane. This is an extreme example; the differences are usually smaller.

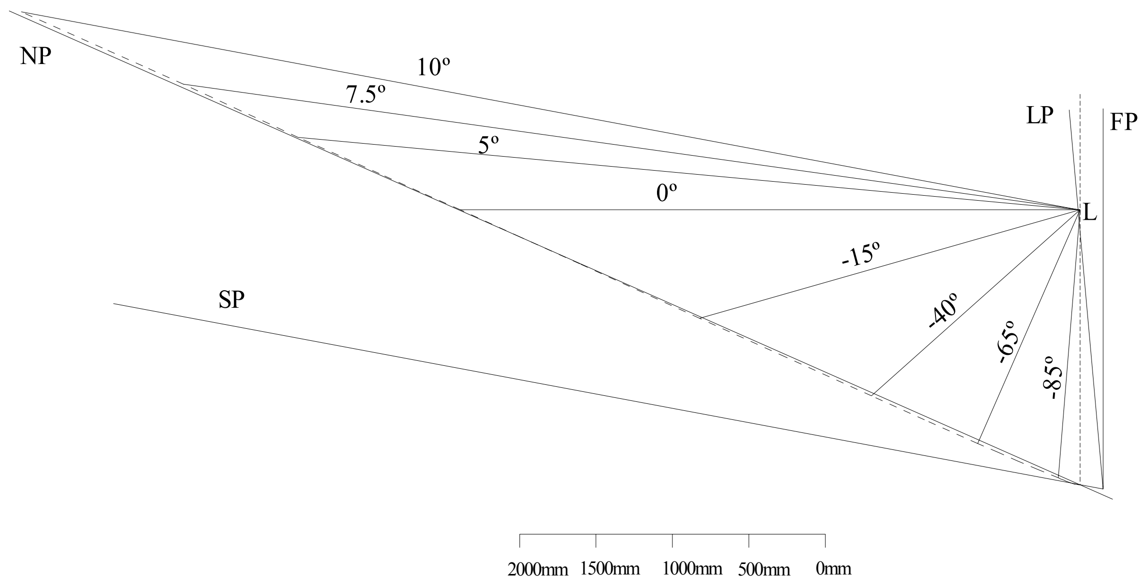


Figure 22: Comparison of near DOF plane (solid) with true curve (dotted).

11 Image control

11.1 Exposure

There are two quantities of prime interest: the luminance, L, of the scene and the exposure, H_f, at the focal plane. The luminance is modified into the exposure by the camera controls, which are the f-number, N and the exposure time t. The exposure is decreased by increasing N, and increased by increasing t. Thus H_f is made smaller by increasing the ratio N^2/t. This means that if one fixes the scene luminance, then the ratio L/H_f increases with N^2/t. In fact, L/H_f is proportional to N^2/t, with the constant of proportionality being fixed by the scene and the film density desired.

Since N and t are under the control of the photographer, and the luminance is fixed by the scene, the photographer needs only two things to make an appropriate exposure: (1) the constant of proportionality, and (2) the value of H_f that will produce the desired density on the film. It turns out that the constant of proportionality can be obtained by a physical argument, so the only thing remaining is to specify H_f in terms of a film.

Now black and white films are characterized by the exposure, H_m, required to produce a density of 0.1 over background and fog. In fact if H_m is measured in cd/m^2, then the ASA film speed is defined as $S = 0.8/H_m$. Thus a fast film, requiring a small value of H_m to produce a 0.1 density, will have a higher speed than will a slower film which needs a larger H_m to produce a 0.1 density. It follows then that one can use S to determine H_f, if one is happy with H_f being defined in terms of a density of 0.1 over background and fog.

This, however, is inconvenient, since measuring the luminance of shadows in a scene can be difficult. Therefore, H_f is usually defined in terms of a larger density, near the middle of the range of interesting densities. For black and white film, the range of usable densities is taken to be 7 stops, hence one needs to choose something like $H_f = rH_m$, where r is approximately 3.5. The upshot is that L/H_f becomes LS/K, where K is a called the *c*alibration constant and is used to encompass both r and the physical constant of proportionality mentioned above. The details may be found in Jacobson(1988), pp 50-52, and pp 264-268. If L is measured in cd/m^2 then $K = 12.5$ for the current ISO standard.

In summary: one has a relationship between the camera controls, N and t and the scene luminance L and film speed S, expressed as

$$\frac{N^2}{t} = \frac{LS}{K},$$

which may seem more familiar in terms of the *e*xposure value EV, defined by

$$EV = log_2(N^2/t) = log_2(LS/K). \tag{52}$$

The ISO standard for color film is defined using the same calibration constant, but since color film involves three films, and since the range of color film is closer to 5 stops, the compromise is not always a happy one.

11.2 Light meters

The calibration problem for color film is apparent for light meter calibration. Newer meters, such as the newer Sekonic's, use the current calibration constant of 12.5. Older meters use other values. The Minolta Spotmeter F uses a calibration constant of 14. The Sekonic manuals state this explicity, but it is easy to obtain for any meter that publishes a table converting EV into luminance. The formula $K = LS/2^{EV}$ may be used with luminance L in cd/m^2 and S the ASA film speed. As mentioned in the previous section, calibration constants determine the correspondence between densities and exposure. Increasing the calibration constant, increases exposure and makes a thinner negative, which some find more appropriate for color film.

Many users find the calibration used by the Minolta Spotmeter F more appropriate for color film, and on occasion have been know to claim that the Sekonic's are miscalibrated. There is clearly a difference between the two calibrations over and above the calibration constant. Changing from $K = 12.5$ to $K = 14$, will decrease the EV by 0.16, however, I find it necessary to decrease the EV by about 0.3 to make the meters match. There also seems to be difference in spectral response of the two meters.

11.3 18% gray cards

Since $log_2(100/18) \approx 2.5$, it follows that in any scene an 18% gray card will be placed 2.5 stops down from the maximum brightness. If the scene has a range of 5 stops, then the middle of the range will be 18% gray, and a lightmeter will place the lowest tone one stop up from H_m. For a 7 stop range, 18% gray will be placed 4.5 stops up from H_m and therefore an 18% gray will not be in the middle of a 7 stop range.

11.4 Sunny 16 rule

The sunny 16 rule states that the correct exposure for bright sunlight should be f/16 at 1/S seconds, where S is the ASA speed. The definiton of exposure value above, one has $N^2/t = SL/K$, and if one takes $t = 1/S$, then one has $N^2 = L/K$. For $N = 16$ this gives L=3200 cd/m^2 which is the proper value for a mid-tone in bright sunlight. Interms of DIN, one has $t = 10^{(DIN-1)/10}$.

11.5 Contrast control – the Zone system

The standard assumes an exposure range of twice 3.6 EV, or about 7 EV. If the luminances in a scene span this or a smaller range, then all will fall on a usable portion of the characteristic curve. For longer ranges, the lower portions of the range will not be captured by the film and both extremes may be lost when printing: the difficulty is less serious for B&W film with its greater latitude. For a range smaller than 7 EV, one may adjust the standard by decreasing the EV. For larger ranges this will not help. For B&W film, one may change the characteristic curve by changing processing times, and this may be used to adjust the range to the range of the paper.

The Zone System is a methodology for controling the contrast of B&W negatives. Adams (1981) assigns 10 zones as shown in Table (4). In practice, the seven zones II through VIII are the only ones of interest. The object is to expose and process film so that the luminance range of the subject is capatured within this seven zone range.

0	Pure paper black
I	Black; virtually indistinguishable from 0
II	Near black; texture but no detail
III	Very dark gray
IV	Dark gray
V	Middle gray
VI	Light gray
VII	Very light gray
VIII	Near white; texture but no detail
IX	Pure paper white

Table 4: Zone definitions

Davis (1999) describes a systemaic methodology for adjusting the contrast for B&W. He uses the subject luminance range, which he designates by *SBR*, as a basis. If ΔE is the range of EV's in the scene to be mapped into a range ΔZ of zones, then he defines $SBR = 7/r$, where $r = \Delta Z/\Delta E$. The assumption being that 7 is a normal range. He multiplies the average gradient, $\overline{G}$ by r to obtain the average gradient required of the characteristic curve in order to place the zones in an acceptable density range for the negative. Experiental effort is required to find suitable processing parameters to achieve $\overline{G}$.

The appropriate exposure is obtained from

$$EV = (loEV) - (loZone - 1.5)/r + 3.6 + \log_2(r), \tag{53}$$

where $loEV$ is the EV to be placed on the zone $loZone$. The argument leading to equation (53) is that since the light meter calibration adjusts EV's by -3.6 to produce mid gray tones on a seven stop scale, one can remove this calibration to obtain a base level corresponding to an exposure which will produce a just noticeable tone: presumably this corresponds to zone I+1/2. Since each zone corresponds to $1/r$ EV units, one must then add $(loZone - 1.5)/r$ to obtain the EV that will produce a tone corresponding to *lo Zone*.

In addition, one must allow for the effect of speed, which changes according to the SBR. If ASA_7 is the ASA speed when $SBR = 7$, then the ASA speed should be adjusted to $r \times ASA_7$.

Davis apparently increments from zone I-1/2, and uses 4 instead of 3.6 to obtain equaion (54),

$$EV = (loEV) - (loZone - 1/2)/r + 4. \tag{54}$$

The difference between equation (53) and (54) is $\log_2(r) - 0.4 - 1/r$, which is negative for $r < 1.9$, resulting in overexposure for developing times that compress the range. The lighter tones will appear to be blocked using Davis's formula.

11.6 Characteristic curves

11.6.1 Graduating function

The following function may be used to describe a characteristic curve:

$$D(x) = F + D_M \sin\left(\frac{\pi}{2}\frac{x - I}{R}\right)^2, \tag{55}$$

where D is density, D_M is the maximum density, x is exposure (log H), I is the intercept, R is the exposure range, and F is base plus fog.

The gradient at x is

$$D'(x) = D_m \frac{\pi}{R} \sin\left(\frac{pi}{2}\frac{x - I}{R}\right) \cos\left(\frac{pi}{2}\frac{x - I}{R}\right),$$

and the average gradient over $[I, I + R]$ is

$$\bar{G} = \frac{D_M}{R} \sin\left(\frac{\pi}{2}\frac{x-I}{R}\right)^2 \Bigg|_{I}^{I+R} = \frac{D_M}{R}, \tag{56}$$

which differs slightly from that used by Davis (1999), where only the center range of densities are used.

11.6.2 Data

	Test 1	Test 3	Test5
F	0.077	0.069	0.056
D_M	0.796	1.792	2.531
I	-2.934	-3.176	-3.220
R	3.805	4.131	3.506
Speed	-2.06	-2.55	-2.77
Davis's Speed	-2.2	-2.6	-2.8
$\bar{G}$	0.21	0.43	0.72
$\bar{G}_c$	0.27	0.57	0.86
Davis's $\bar{G}_c$	0.31	0.56	0.88
SBR	9.8	5.8	3.9
Davis's SBR	10.7	5.9	3.8

Table 5: Fitted parameters

The data in table (6) was copied from page 47 of Davis (1999), and fitted with equation (55). Figure (23) shows a plot of the fitted values. The fit is essentially a perfect fit, suggesting that the graduating function captures the essence of the underlying mechanism. Similar plots with other films show the same good agreement.

Table (5) gives the fitted parameters plus other derived estimates. Film speed is obtained as the log H value corresponding to a density of 0.1+F. The $\bar{G}$ is calculated from equation (56), and $\bar{G}_c$ is the average gradient obtained from the center range of densities from 0.1+F to DR+0.1+F using equation (55), where DR is the useable density range which is approximately equal to the paper's SI: here the DR is taken as 1.0. The subject luminance range, SBR, defined by Davis is $DR/(\bar{G}_c \log(2))$.

It is important to note that the maximum density predicted by the graduating function is M, which in the case of Test 1 is less than DR+0.1+F. In this case $\bar{G}_c$ is calculated using the density range from 0.1+F to M. Since the graduating function maximizes at this point, it suggests that the development process will not achieve higher densities, and than greater exposures will produce a decreasing density solarizaiton. The SBR is therefore taken as $M/\bar{G}_c$

Fitted Values

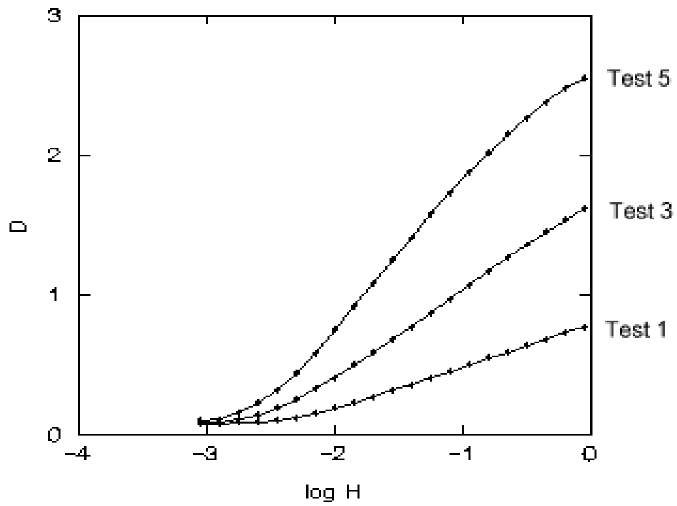


Figure 23: Fitted values

12 Perspective and normal lenses

Without the ability to assess distance using binocular vision, the visual system must rely on other clues: the principal one being angular size. The assumption that the angular size of an object is proportional to distance is not a bad one. The ratio of object height, h, to distance, s, is $h/s = (h/f)[m/(1+m)]$ which is approximately $m(h/f)$ for small m, where f is focal length, and m is magnification. Thus twice $m(h/f)$ means either that the object is twice the size $m(2h/f)$ or twice as close $2m(h/f)$, since $2/s \approx 2m/f$. For small angles, one has $\arctan h/s \approx h/s$ and thus the proportionality also applies to angles.

The human eye is not a camera, but it does obey the laws of optics, and it is interesting to discuss it as if it were a camera. Such a camera would have a focal length of about $1''$ and the image projected on the retina would also be $1''$. It follows that the angle of view, AOV, is $2\arctan(1/2) = 53°$.

Let D_n be the nearest distance of clear vision, which seems to be in the neighborhood of $10''$ for normal eyes. The magnification is thus $m = 1/D_n$, and an object filling the retina will be of size D_n. Since m is small, the proportionality holds for the entire practical range of the visual system: for all objects from the nearest focus to the most distant. Cameras are not limited to small m, and nearby objects which violate proportionality, can be brought into focus, enabling pictures in situations to which the eye is unaccustomed.

log H	Test1 D	Fit 1	Test 3 D	Fit 3	Test5 D	Fit 3
-3.05	0.08	0.08	0.09	0.07	.10	0.07
-2.9	0.08	0.08	0.09	0.09	0.11	0.11
-2.75	0.08	0.09	0.11	0.12	0.16	0.17
-2.6	0.09	0.09	0.14	0.15	0.23	0.25
-2.45	0.10	0.11	0.19	0.20	0.32	0.34
-2.3	0.12	0.13	0.25	0.26	0.44	0.46
-2.15	0.15	0.16	0.33	0.33	0.58	0.59
-2	0.19	0.19	0.41	0.40	0.75	0.74
-1.85	0.23	0.23	0.50	0.49	0.92	0.90
-1.7	0.27	0.27	0.59	0.58	1.08	1.06
-1.55	0.32	0.31	0.68	0.67	1.25	1.23
-1.4	0.36	0.36	0.77	0.77	1.41	1.40
-1.25	0.41	0.40	0.87	0.87	1.58	1.57
-1.1	0.45	0.45	0.97	0.97	1.73	1.73
-0.95	0.50	0.50	1.07	1.07	1.88	1.89
-0.8	0.55	0.55	1.17	1.18	2.01	2.03
-0.65	0.59	0.60	1.27	1.27	3.15	2.17
-0.5	0.64	0.64	1.36	1.37	2.27	2.29
-0.35	0.68	0.69	1.45	1.46	2.38	2.39
-0.2	0.73	0.73	1.54	1.54	2.48	2.47
-0.05	0.77	0.76	1.62	1.61	2.55	2.53

Table 6: Observations and fitted values

To appear normal, a picture held at D_n should produce the same angles for objects as does the image that forms on the retina. An image occupying a proportion, α, of the picture will have an angle of $53\alpha°$. If a camera lens has this AOV, then an object occupying the proportion α will also have angle $53\alpha°$, and thus appear normal on a print. It follows then that normal lenses are those whose focal lengths are approximately equal to the diagonal of the film size.

A lens with a smaller than normal focal length will have an AOV greater than 53°, making an object occupy a smaller angle and thus appearing farther away. A larger focal length does the opposite, making distant objects appear closer: the consequence of this is that two objects which to the naked eye appear X units apart, will appear to be to be $X53/A_L$ units apart, where A_L is the AOV for the longer lens. For shorter lenses with A_S, the AOV, the situation is reversed, so the objects appear to be $XA_S/53$ units apart: the separation appears greater than the naked eye expects. In addition, short lenses are more frequently used for nearby objects for which proportionality does not hold, producing even greater exaggeration. It should be noted, of course, that such nearby objects, even when they cannot be brought into focus, still appear reasonable to the eye because binocular vision provides triangulation information to the visual system.

13 Digital

13.1 Scanning and resolution

The aim of scanning is to extract that amount information from the source material required by the output material. How much is this? Obviously it varies with the purpose, but for photographs, it need not be more than the human eye can use. The minimum distance of distinct vision is about 25 cm. Which means that most people cannot focus on closer objects. In bright light the human eye at this distance can just resolve 5 l/mm which sets a limit on the print resolution required. What is the corresponding resolution on the film?

There are two parts to the problem: (1)Scan to digital, (2) Digital to print.

(1) A careful examination of scans a USAF test target, suggests about 3 dots seems to be the minimum number required to separate lines. A line on these charts is the extent of a black and white bar, which is the distance between the frontiers of one line and its successor. With two dots, the line may be missed because both dots fall in the black or in the white, while with 3 there will be at least one on each color. It follows that for a scanning density of R dpi, the captured film resolution r_f is about $R/3$ l/in, or about $R/75$ l/mm.

(2) The density, R, must be determined from the print requirements. Now 5 l/mm, the resolution of the human eye, corresponds to 127 l/in, but printers need 2 input dots to show a USAF test target line, this means that about 250 dpi must be sent to the printer. As confirmation, tests using photographs of a USAF test target with the Epson Photo EX shows that this printer requires a value between 200 and 240 dpi. The 240 dpi is excellent, and the 200 dpi is almost as good, but a slight degradation can be detected. Large commercial printers often require 300 dpi. For convenience, let us choose 225 dpi as the required dot density for a printer.

The total number of dots required by a printer is $T = 225 \times d_p = R \times d_f$, where d_p and d_f are corresponding dimensions of the print and the film. From this, one has $R = T/d_f$, and thus the captured r_f becomes $r_f = R/75 = (T/d_f)/75 = (225d_p/d_f)/75 = 3M$ in l/mm, where $M = d_p/d_f$. If one chooses 2 dots instead of 3 in paragraph (1), then $r_f = 4M$ or $5M$. If one sends 300 dpi to the printer, as one would to with a LightJet 5000, then $r_f = 4M$.

Taking $r_f = 3M$, the captured r_f is shown in Table 7 for several film and print sizes. Appropriate scan rates for each film size and the maximum on-film captured r_f is also shown.

	4x5	8x10	16x20	R	R/75
35	12	24	51*	3000	40
6x7	6	12	24	2000	27
4x5	3	6	12	1000	13

Table 7: Captured film resolutions in l/mm given print size
* Not possible with available scanning rate

One sees, for example, that the minimum captured detail in an 8x10 digital print from a 4x5 transparency is about 6 l/mm. This is less than the minimum resolution for a conventional print of about 10 l/mm. The difference is traceable to the fact that 3 dots are needed by the scanner for a digital print. The loss can often be compensated by sharpening the edges with unsharp masking.

It is useful to note that four times the 4x5 r_f is equal to the 35 mm r_f. Therefore given a fixed print size, the resolutions for all formats represent the same degree of detail in the subject: e.g. an object just resolved in an 8x10 print from a 35 mm slide will be just resolved in an 8x5 print from a 4x5. This is not true for a 16x20 print, where there is a difference between 35 mm and 4x5 because the 35 mm r_f is limited to 40 l/mm (= 3000/75) instead of 48 l/mm.

As a caution, there is always much more resolution in a film than may be captured in a scan. Figure 17 on page 33, shows a stack of dollars which were photographed and scanned at 3000 dpi – it is printed here only to show the test setup, since the resolution is low in this printing. The photo used was 132-2 in my files. The resolution of the film is near 60 or so l/mm, since the major fine lines at 45 l/mm are clear with a 20 power loupe, but the lines in the leaves at about 72 l/mm are not distinguishable[12].

In the scanned image, the major fine lines can be seen against a black background, but they cannot be separated when near each other as in the wreaths, nor can the lines in the leaves be separated. I conclude that one needs more than two dots to make a line clearly distinguishable. If one requires 3 dots instead of 2, then the resolution from a 3000 dpi scan is 3000/150 = 20 l/mm, which seems reasonably consistent with these observations. Scanned photographs of the USAF chart confirm that 3 dots are needed at the highest detectable frequency.

13.2 Digital cameras

The arrays in digital cameras are much smaller than film sizes. The commonly used ones are shown in Table 8. The dimensions have been rounded to the nearest millimeter. The names 1/3 inch, etc. are a holdover from previous technology and do not represent the actual size of the arrays.

	height	width	pixels
1/3 inch	4	5	480x640
1/2 inch	5	6	1000x1300
2/3 inch	6	9	1400x1700
large	18	28	2000x3000

Table 8: Approximate digital camera array sizes in mm

[12]Comparing the line width with that of an Edmund chart shows the major lines are about 5 l/mm, and the ones on the leaves perhaps 8 l/mm. A 150 mm lens was used at a distance of 1500 mm, giving a multiplying factor of 9, so $9 \times 5 = 45$.

	3x5	5x7	8x10	R	R/75
1/3 inch	57*	95*	152*	3200	43
1/2 inch	45	76*	122*	5000	67
2/3 inch	38	63	102*	6000	80
large	13	21	34	2820	38

Table 9: Captured digital array resolutions in l/mm given print size
* Not possible with available array dimensions

A table of captured resolutions for these cameras is shown in Table 9. The R in this table is the number of pixels per inch for the shorter dimension of the array. It may be seen that only the large array is adequate for 8x10 prints. A limited test seems to confirm the accuracty of this table. Tests on the Olympus C-2500L with a 2/3 inch array produced 60 to 68 l/mm depending on the focal length used. This agrees with the captured resolution in Table 9 of 63 l/mm.

14 Miscellany

14.1 Fractional stops

The amount of light admitted through the aperture is of great importance, as early photographers quickly learned. It may be controlled by the duration of the exposure and by the lens aperture. About 1880 according to (Kingslake, 1989) it became customary to designate a lens by its f-number. Iris shutters were also introduced about 1880 (*ibid*), and markings on the lenses were likely to have been in terms of f-numbers. Sequences of stops and f-numbers were proposed at the turn of the century, but the present standard sequence of f-numbers {1, 1.4, 2, 2.8, 4, ...} is fairly recent.

There is not a lot that can be said in justification of this standard sequence for a practical photographer. It requires an act of memory to use, and it increases as the light admitted decreases, which is contrary to what one wants. One way to think of it is as a measure of the luminance of the object, in which case a small opening and large f-number represents a bright object, but the odd sequence is still something to remember.

A rational scale of values corresponding to aperture diameters[13] is the exposure value, EV, scale. From equation (52), one can see the relationship between EV and the f-number, N by setting the time t to unity: it is $N^2 = 2^{EV}$, or $EV = 2log_2(N)$.

The EV value 2.5 corresponds to a point between $f/2$ and $f/2.8$. It is this

[13]The blades of iris diaphragms are curved to produce an aperture scale with equidistant spacings, (Ray,1994). The movement may not be completely uniform for intermediate stops, since the opening is a polygon, not a circle. In general fractional stops on large format lenses should not be relied on to less than 1/3 stop and probably not to 1/2 stop.

"fractional" value that appears on digital light meters[14]: e.g. a meter reading might be 1.4₅ with the "5" in smaller type than the "1.4". Useful notations for the printed page are 1.4_5, and $1.4 + 1/2$. Cameras with automatic diaphragms can be expected to adjust their apertures to this scale, at least to a 0.1 step (Ray, 1994).

The amount of light actually admitted through the diaphragm is not, however, %50 more at $EV = 2.5$ than at $EV = 2.0$. From equation (52), one has $2^{EV} = LS/K$. Thus $2^{EV+0.5} = (LS/K)2^{0.5} = (LS/K)\sqrt{2} \approx (LS/K)1.41$; thus the increase is %41 not %50.

When magnification, m, is not zero, one may desire the number of stops associated with the bellows factor. If V is the f-number when focused at infinity, then $U = V/(1+m)$ is the f-number allowing for the bellows extension. Suppose EV_1 is the exposure value corresponding to V, that is $2^{EV_1} = V^2$, then $U^2 = 2^{EV_1}/(1+m) = 2^{EV_1 - log_2(1+m)}$. Thus one opens up $log_2(1+m)$ stops[15] from V to obtain the correct f-number U.

14.2 Stopping Motion

Let c be the diameter of the circle of confusion, and s the corresponding subject size at distance u for a lens with focal length f, then

$$\frac{c}{s} = \frac{f}{u-f},$$

and if an object moves at a rate r then $s = r/T$, where $1/T$ is the shutter time in seconds, one finds that the shutter speed necessary to stop this motion is

$$T = \frac{rf}{c(u-f)} = m\frac{r}{c},$$

where m is magnification and r is in millimeters per second.

If r is in mph, u in meters, and f and c in millimeters, then

$$T = \frac{447.04rf}{c(1000u - f)}.$$

It may also be useful to cite the distance the image of a point moves during $1/T$ seconds. This is given by

$$\frac{mr}{cT},$$

where the distance is in units of c.

[14]See the Minolta Spotmeter F instruction manual, or the tables in the Sekonic Zoom Master L-508, or L-608.

[15]Remember that a decrease in EV corresponds to a decrease in f-number, and an increase in aperture size.

14.3 Bellows Factor

The inverse square law applies to bellows movements: the intensity of the light falling on a surface is inversely proportional to the square of its distance from the lens. Thus a bellows with extension equal to the focal length of the lens delivers $\frac{1}{4}$ the light, or exposure, as one with zero extension.

If e is the extension of the bellows, then the bellows factor is defined as

$$BF = \left(\frac{f+e}{f}\right)^2 = (1+m)^2,$$

where m is magnification.

The time value ($1/sec$) is multiplied by BF to adjust for the decrease in light, or the f-number can be divided by $\sqrt{BF}$.

An easy procedure is to divide $f+e$ and f by something that brings them into the conventional f-number range and then to count the stop differences: thus, $e = 250$ with $f = 200$ produces the 45 and 20 f-numbers, which is a bit more than 2 stops. This may be compared with the actual value of 2.3 stops given by $\log_2(BF)$.

14.4 Stops as distances

The inverse square law gives the ratio $(d_2/d_1)^2$ for light intensity at two distances d_1 and d_2, and the number of stops is the base 2 log of this ratio; hence $2log_2(d_2/d_1)$ is the change in stops.

14.5 AOV

The angle of view, A, is the angle subtended at the center of the lens by either the object field or the image extent. Simple trigonometry shows:

$$A = 2\tan^{-1}\left(\frac{d}{2f(1+m)}\right),$$

where d is the image extent, m the magnification, and f the focal length.

It is useful to note that this may be rearranged as,

$$1+m = \frac{d}{2f\tan(A/2)},$$

which shows that if f is scaled by the same scaling as d when format is changed, then magnification is unaffected for a constant AOV. The camera must be moved to accommodate this, since

$$v = f(1+m) = \frac{d}{2\tan(A/2)},$$

changes with d.

14.6 Film flatness and vibration effects

The effect of a lack of film flatness or vibration may be assessed in terms of changes in the depth of focus. Equation (34) may be used with $c = 1/L$:

$$L = \frac{2N(1+m)}{\delta},$$

where δ is the depth of focus along the rail, and L is lines per millimeter. The differencial of δ with respect to L becomes

$$d\delta = -\frac{\delta}{L} dL = -\delta c\, dL,$$

which is the amount of film movement which degrades the image by one line per millimeter – the smaller the $|d\delta|$ the more sensitive. One should probably use half of this, since the film backing is presumably fixed.

This suggests that film movement disturbs the image at the exact focus more than at the DOF limits.

14.7 Hand holding a camera

Taking $1/f$ as the exposure time is a common rule for hand holding. It has some justification: First assume that halving the time of exposure will halve the size of the blur. Fix the magnification so that the subject will always image the same size on the film. Suppose that vibrations from a lens of focal length f, can be damped by an exposure time of $1/f$. This means that a point in the subject will image as a circle of confusion, c. It follows that a $2f$ lens puts the image twice as far from the lens, and with the same amount of vibration, the image will be of size $2c$.Hence one should cut the vibrations in half by using $1/(2f)$ as the time of exposure to obtain an image of size c.

A test must be run to justify $1/f$ in the first place. It is likely that a parameter will be needed to reflect individual variation.

14.8 Finding the 1:1 focus point

Since $u + v = \frac{u^2}{u-f}$ which has a minimum at $u = 2f$, one sees that the 1:1 focus point is attained when $u + v$ is a minimum. In practice one could use a measurement from a mark on the rear standard to a mark on something that moves with the subject and vary the subject distance until the minimum in-focus distance is found.

14.9 Miscellaneous equations

Magnification as a function of $nearDOF$.

$$m = \frac{Nc(f-\Delta)}{f\Delta} + \sqrt{\left(\frac{Nc(f-\Delta)}{f\Delta}\right)^2 + \frac{Nc}{\Delta}}, \tag{57}$$

where $\Delta = u - Z_n$ is $nearDOF$.

Focal length as a function of DOF.

$$f = \left(\frac{Nc}{m}\right)\sqrt{\frac{DOF}{DOF - 2NcM}}, \tag{58}$$

where $M = (m+1)/m^2$.

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